p. 260 #22. If \( f(x) = -(x - 2)^2 \) then \( f'(x) = -2(x - 2) \) and hence \( f'(2) = 0 \). Note that \( f(2) = 0 \) and \( f(x) < 0 \) for all \( x \neq 2 \). So \( x = 2 \) is a local maximum.

p. 260 #26. If \( f(x) = -(x - 1)^5 \) then \( f'(x) = -5(x - 2)^4 \) and hence \( f'(1) = 0 \). But if \( x > 1 \) then \( f(x) < 0 \) and if \( x < 1 \) then \( f(x) > 0 \). Since \( f(1) = 0 \) is not greater than all of the points around it or less than all of the points around it, it is neither a local maximum nor a local minimum.

p. 260 #30. At \( x = 2 \) and \( x = -2 \), \( f(x) = 0 \). For all other points, \( f(x) < 0 \). So \( f(x) \) has maxima at \( x = 2 \) and \( x = -2 \). However, at \( x = 2 \), the derivative would be

\[
\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}.
\]

We check whether this limit is defined by examining it from the right and from the left. Note that, for \( h \) slightly positive, \((2 + h)^2 - 4\) is slightly greater than zero. So \(|(2 + h)^2 - 4| = (2 + h)^2 - 4\), and hence \( f(2 + h) = -(2 + h^2 - 4) \). Also, note that \( f(2) = 0 \). So the limit from the right is

\[
\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{-((2+h)^2-4)-0}{h} = \lim_{h \to 0^+} \frac{-(4+4h+h^2)-0}{h} = \lim_{h \to 0^+} \frac{-4h-h^2}{h}.
\]

Canceling an \( h \) from the numerator and the denominator gives

\[
\lim_{h \to 0^+} -4 - h = -4
\]

Now, for the limit from the left, if \( h \) is slightly negative, \((2+h)^2-4\) is also negative. So \(|(2+h)^2-4| = ((2+h)^2-4)\) and hence \( f(2+h) = (2+h)^2-4 \). Thus, the limit from the left is

\[
\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{-(2+h)^2-4-0}{h} = \lim_{h \to 0^-} \frac{-4-4h-h^2}{h} = \lim_{h \to 0^-} 4 + h = 4.
\]

Since these are unequal, the derivative is not defined at \( x = 2 \).

For \( x = -2 \), the reasoning is the same. To evaluate the limit from the right, note that, for \( h \) slightly negative, \((-2+h)^2-4\) is positive. So \(|(-2+h)^2-4| = (-2+h)^2-4\) and hence \( f(-2+h) = -((-2+h)^2-4) \). Also, \( f(-2) = 0 \). So the limit from the left is
\[ \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h \to 0} \frac{-(2+h)^2 - 4}{h} \]
\[ = \lim_{h \to 0} \frac{-4+4h+h^2 - 4}{h} \]
\[ = \lim_{h \to 0} \frac{-4h-h^2}{h} \]
\[ = \lim_{h \to 0} -4 - h \]
\[ = -4. \]

If \( h \) is slightly positive, \((-2 + h)^2 - 4\) is negative. So \(|(-2 + h)^2 - 4| = -((-2 + h)^2 - 4)\) and hence \(f(-2 + h) = (-2 + h)^2 - 4\). Thus, the limit from the right is

\[ \lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{((2+h)^2 - 4)}{h} \]
\[ = \lim_{h \to 0^+} \frac{4+4h+h^2 - 4}{h} \]
\[ = \lim_{h \to 0^+} \frac{4h+h^2}{h} \]
\[ = \lim_{h \to 0^+} 4 + h \]
\[ = 4. \]

Since the limits are unequal, the derivative is not defined at \( x = -2 \).

p. 260 #32.) By examining the graph, we see that \((2,0)\) and \((-2,0)\) are local and global maxima. Also, \((3,-1)\) and \((-3,-1)\) are local minima, and \((0,-2)\) is a local and global minimum.

p. 260 #38.) Note that \( f(-1) = 0 \) and \( f(2) = 0 \). So, by MVT, there is a point \( c \) between \(-1\) and \( 2 \) such that
\[ f'(c) = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{0}{3} = 0. \]

We can find \( c \) by taking
\[ f'(x) = 2x - 1 = 0. \]

This means that
\[ 2x = 1 \]
or
\[ x = \frac{1}{2}. \]
p. 261 #48a.) To find the average velocity, we know that $s(0) = 0$ and $s(10) = 10$. So

$$\frac{\Delta s}{\Delta t} = \frac{s(10) - s(0)}{10 - 0} = \frac{10 - 0}{10 - 0} = 1 \text{ km/sec.}$$

#48b.) The instantaneous velocity is the derivative $s'(t) = \frac{1}{5} t$.

#48c.) We wish to find when $s'(t) = 1$. So

$$\frac{1}{5} t = 1$$

$$t = 5.$$  

p. 270 #6.) First, note that $16 - x^2$ is negative when $x$ is between $-4$ and $4$, and positive otherwise (i.e. when $-5 \leq x < -4$ or $4 < x \leq 8$. So

$$f(x) = |16 - x^2| = \begin{cases} 16 - x^2 & \text{if } -5 \leq x \leq -4 \text{ or } 4 \leq x \leq 8 \\ -(16 - x^2) & \text{if } -4 < x < 4 \end{cases}$$

Using this, we take the derivative of each of these functions to see that

$$f'(x) = \begin{cases} -2x & \text{if } -5 \leq x < -4 \text{ or } 4 < x \leq 8 \\ 2x & \text{if } -4 < x < 4 \end{cases}$$

Of course, $f(x)$ is not differentiable at $x = \pm 4$ (since the graph has a corner at each of those places). Clearly, if $f'(x) = 0$ then $x = 0$. Thus, our candidates for maxima and minima are $x = 0$ (since $f'(x) = 0$), $x = \pm 4$ (since $f'(x)$ is undefined there), and $x = -5$ and $8$ (the endpoints).

Examining these on the graph, we see that $(0, 16)$, $(8, 48)$, and $(-5, 9)$ are local maxima. We know that the function has a global maximum by the extreme value theorem (because the interval is closed), so this global maximum is merely the largest of the local maxima; hence, $(8, 48)$ is the global maximum.

Similarly, the points $(4, 0)$ and $(-4, 0)$ are local minima. Since they are clearly the lowest points on the graph (an absolute value can’t be less than zero), they are also both global minima.

Also, from our derivatives above (or from the graph), we can see that the function is increasing (i.e. derivative is positive) when $-4 < x < 0$ and $4 < x < 8$ and decreasing everywhere else.

p. 270 #22.) Here, we have $f(x) = xe^{-x}$. Using the product rule yields

$$f'(x) = e^{-x} - xe^{-x}$$

$$f''(x) = -e^{-x} - e^{-x} + xe^{-x}.$$  

To find inflection points, we set the second derivative equal to zero. So
\[-e^{-x} - e^{-x} + xe^{-x} = 0\]
\[-2e^{-x} + xe^{-x} = 0\]
\[e^{-x}(-2 + x) = 0.\]

So $e^{-x} = 0$ or $-2 + x = 0$. The former never happens; the latter happens when $x = 2$. So $x = 2$ is the only possible inflection point. To ensure that it is actually an inflection point, check a point to the left and to the right. Note that $f''(0) = -2$ and $f''(3) = e^{-3} > 0$. Since the signs are different, $x = 2$ is indeed an inflection point.

p. 270 #27.) If $f(x) = \frac{2}{3}x^3 - 2x^2 - 6x + 2$ for $-2 \leq x \leq 5$ then

\[f'(x) = 2x^2 - 4x - 6,\]
\[f''(x) = 4x - 4.\]

To find possible extrema, we first set $f'(x) = 0$. Solving this using the quadratic formula gives $x = 3, -1$. If we check the second derivative, we see that $f''(3) = 8 > 0$ and $f''(-1) = -8 < 0$. So $x = 3$ is a local minimum and $x = -1$ is a local max. Additionally, if we examine the endpoints on the graph, we see that $x = 5$ is a local max and $x = -2$ is a local min.

For inflection points, we see that $f''(x) = 0$ when $x = 1$, which is therefore our only possible inflection point. Note that, for $x > 1$, $f''(x) > 0$, and for $x < 1$, $f''(x) < 0$. Thus, $x = 1$ is the inflection point, and the graph is concave up when $x > 1$ and concave down when $x < 1$.

p. 270 #34.) Again, we find the first and second derivative:

\[f'(x) = \frac{2x}{(1+x^2)^3} - \frac{(2x)(2x)}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}.\]

To find the local extrema, we note that $f'(x) = 0$ when $x = 0$. This is the only possible extremum, and, since $f''(0) = 2 > 0$, it is a minimum. If $x > 0$ then the function is clearly increasing ($f'(x) > 0$); otherwise, the function is decreasing.

For inflection points, we see that $f''(x) = 0$ when

\[2 - 2x^2 = 0\]
\[1 = x^2\]
\[x = \pm 1.\]

If $x > 1$ or $x < -1$, $f''(x) < 0$ (and the graph is concave down), while if $-1 \leq x \leq 1$, $f''(x) > 0$ (and the graph is concave up). Since concavity changes
at \( x = 1 \) and \( x = -1 \), both are inflection points.

p. 271 #36a.) Since the degree of the denominator is greater than that of the numerator, the limit as \( x \to \infty \) or \( -\infty \) is zero.

p. 271 #36b.) If \( |x| \) is slightly smaller than one (i.e. \( x < 1 \) or \( x > -1 \)) then the limit as \( x \to 1 \) of \( f(x) \) is positive; hence, \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \infty \). By the same reasoning (but in the opposite direction), \( \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = -\infty \).

p. 271 #36c.) Taking the derivative gives

\[
f'(x) = -2(-1)(2x)(x^2 - 1)^{-2} = \frac{4x}{(x^2 - 1)^2}.
\]

The bottom is always positive (or zero), the top is positive when \( x < 0 \) or negative otherwise. So the function is increasing when \( x < 0, x \neq -1 \), and it is decreasing when \( x > 0, x \neq 1 \). \( x = 0 \) is therefore a local maximum.

p. 271 #36d.) Taking the second derivative,

\[
f''(x) = \frac{4(x^2 - 1)^2 - (4x)(2(x^2 - 1)2x)}{(x^2 - 1)^4} = \frac{4x^2 - 4 - 16x^2}{(x^2 - 1)^3} \cdot \frac{x^2 - 1}{(x^2 - 1)^2}.
\]

This is zero when the top is zero, which is when \( -4 = 12x^2 \); this never happens. So there are no inflection points. Note that \( f''(x) > 0 \) when \( x > 1 \) or \( x < -1 \) and \( f''(x) < 0 \) when \( -1 < x < 1 \). So the concavity acts accordingly.

p. 296 #6.) If we let \( a \) and \( b \) be the sides of a right triangle (other than the hypotenuse), then we wish to maximize the area

\[
A = \frac{1}{2} ab
\]

given that

\[
a^2 + b^2 = 4^2.
\]

Let us eliminate one of the variables by writing it in terms of the other one:

\[
a = \sqrt{4^2 - b^2}.
\]

We plug this into the equation \( A \):

\[
A = \frac{1}{2}(\sqrt{16 - b^2})b.
\]
Now, to find the maximum of $A$, take the derivative equal to zero:

$$A' = \frac{1}{4}(16 - b^2)^{-\frac{1}{2}}(-2b) + \frac{1}{8}\sqrt{16 - b^2} = 0$$

$$A' = -\frac{1}{2}b^2(16 - b^2)^{-\frac{1}{2}} + \frac{1}{8}\sqrt{16 - b^2} = 0.$$ 

Multiplying both sides by $2\sqrt{16 - b^2}$ gives

$$-b^2 + (16 - b^2) = 0$$

$$8 = b^2$$

$$b = \pm\sqrt{8}.$$ 

To check that this is a maximum, we use the second derivative. So

$$A'' = -b(16 - b^2)^{-\frac{3}{2}} + \frac{1}{4}b^2(16 - b^2)^{-\frac{3}{2}}(-2b) + \frac{1}{8}(16 - b^2)^{-\frac{1}{2}}(-2b).$$ 

Note that, for $b = -\sqrt{8}$, $A'' > 0$, while for $b = \sqrt{8}$, $A'' < 0$. Thus, the only maximum is $b = \sqrt{8}$. Plugging back into $A$, we get that $A = 4$.

p. 296 #10.) We wish to maximize

$$A = 2xy$$

with the condition that

$$y = \sqrt{4 - x^2}.$$ 

Plugging the latter into the former gives

$$A = 2x(\sqrt{4 - x^2}).$$ 

We take the derivative of $A$ and set it equal to zero, giving

$$A' = (4 - x^2)^{-\frac{3}{2}}(-2x)x + 2\sqrt{4 - x^2} = 0$$

$$A' = -2x^2(4 - x^2)^{-\frac{3}{2}} + 2\sqrt{4 - x^2} = 0.$$ 

Multiplying both sides by $\frac{1}{2}\sqrt{4 - x^2}$ gives

$$-x^2 + (4 - x^2) = 0$$

$$2 = x^2$$

$$x = \pm\sqrt{2}.$$ 

To check that this is a maximum, we use the second derivative. So

$$A'' = -4x(4 - x^2)^{-\frac{1}{2}} + x^2(4 - x^2)^{-\frac{3}{2}}(-2x) + (4 - x^2)^{-\frac{1}{2}}(-2x).$$
If \( x = \sqrt{2} \) then \( A'' \) is negative, while if \( x = -\sqrt{2} \) then \( A'' \) is positive. Thus, the maximum is at \( x = \sqrt{2} \), at which point \( A = 4 \).

p. 307 #6.) Plugging in \( x = 0 \) gives \( \frac{0}{0} \). So we can use L'Hopital's Rule:

\[
\lim_{x \to 0} \frac{3-\sqrt{2x+9}}{2x} = \lim_{x \to 0} \frac{-\frac{1}{2}}{\frac{1}{2}.}
\]

If we let \( x \) go to zero then the above is

\[
\lim_{x \to 0} \frac{-\frac{1}{2}}{2} = -\frac{1}{6}.
\]

p. 307 #16.) If we let \( x = 0 \) in our expression then we get \( \frac{0}{0} \). So we can use L'Hopital's Rule:

\[
\lim_{x \to 0} \frac{e^x-1-x^2}{3x^2} = \lim_{x \to 0} \frac{e^x-1-x}{3x}.
\]

Letting \( x \) go to zero still gives \( \frac{0}{0} \). So we use L'Hopital's rule again:

\[
\lim_{x \to 0} \frac{e^x-1-x}{3x} = \lim_{x \to 0} \frac{e^x-1}{6x}.
\]

Again, letting \( x \) go to zero still gives \( \frac{0}{0} \). So we use L'Hopital's rule once more:

\[
\lim_{x \to 0} \frac{e^x-1}{6x} = \lim_{x \to 0} \frac{e^x}{6} = \frac{1}{6}.
\]

p. 307 #22.) To make this into a form which can use L'Hopital’s Rule, we must convert this to a fraction which goes to \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \) as \( x \to \infty \). This is accomplished by considering \( \frac{x^2}{e^x} \). Now, we can use L'Hopital’s Rule

\[
\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x}.
\]

Since this still goes to \( \frac{\infty}{\infty} \), we use L'Hopital again:

\[
\lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0.
\]