SAT[®]/ACT[®] Math

And Beyond: Problems Book

A Standard High School Workbook

First Edition

Qishen Huang, Ph.D.

This book helps you

 $\sqrt{\text{Score highly on SAT/ACT Math section,}}$



 $\sqrt{}$ Win in high school math contests.

ISBN-10: 0-9819072-0-2

ISBN-13: 978-0-9819072-0-8

© Copyright 2008 by Qishen Huang.

It is unlawful for anyone to incorporate any part of the content into his works without the author's permission. Questions for the author should be sent to *GoodMathBook@Yahoo.com*.

Limit of Liability/Disclaimer of Warranty: The author makes no representations or warranties with respect to the accuracy or completeness of the contents of this book and specifically disclaims any implied warranties of anything for a particular purpose. The author shall not be liable for any loss of profit or any other commercial damages, including but not limited to special, incidental, consequential, or other damages.

ACT is a registered trademark of ACT, Inc. SAT is a registered trademark of the College Entrance Examination Board. Both companies were not involved in the creation and marketing of the book.

ISBN-10: 0-9819072-0-2 ISBN-13: 978-0-9819072-0-8

10900 Stonecutter Place, Gaithersburg, MD 20878-4805, USA.

Preface

I performed most of groundwork for this workbook while my oldest son was in high school. With the book, I have three relevant goals for him.

1. To get a perfect or near perfect score on the SAT Math section.

It is getting harder for high school students to gain acceptance to a decent college these days because of the large number of applicants across the country. Students can read the article *Applications to Colleges Are Breaking Records* by Karen W. Arenson dated January 17, 2008 in The New York Times. A top math score gives students an edge against competition.

2. To build a solid foundation for college level math.

With the growing number of problems that need to be solved by advancements in science and technology, current and future generations cannot afford to have a weak foundation in mathematics. Johns Hopkins University Mathematics Professor W. Stephen Wilson gave his 2006 calculus class the same test his 1989 class had taken, and the 2006 students were wiped out by the old class. A stitch in time saves nine.

3. To be competitive internationally in math.

According to the article U.S. Leaders Fret Over Students' Math and Science Weaknesses by Vaishali Honawar of Education Week, US high school students had lower math score than any other developed country. Rep. Vernon Ehlers of Michigan declares it a steadily worsening crisis. Central to the crisis is a popular culture that doesn't value math and science.

This problems book will be updated from time to time. A detailed Solutions Manual is available. Relevant inquires should be sent to *GoodMathBook@Yahoo.com*..

Contents

1	Tips on Math Homework	1
2	Algebraic Expressions: Basic	3
3	Algebraic Expressions: Intermediate	7
4	Rational Expressions	9
5	Linear Relations: Basic	13
6	Linear Relations: Intermediate	18
7	Linear Relations: Advanced	21
8	Word Problems: Basic	23
9	Word Problems: Intermediate	25
10	Word Problems: Advanced	27
11	Geometry: Basic	29
12	Geometry: Intermediate	34
13	Geometry: Advanced	39
14	Radicals	44
15	Exponentials: Basic	47
16	Exponentials: Intermediate	52
17	Exponentials: Advanced	54
18	General Functions	57
©	Qishen Huang	4

19 Inverse Functions	60
20 Quadratic Functions: Basic	62
21 Quadratic Functions: Intermediate	66
22 Quadratic Functions: Advanced	68
23 Polynomial and Rational Functions	70
24 Radical Equations and Functions	74
25 Circles	76
26 Ellipses	79
27 Hyperbolas	82
28 Sequences: Basic	86
29 Sequences: Intermediate	91
30 Sequences: Advanced	93
31 Trigonometry: Basic	94
32 Trigonometry: Intermediate	103
33 Trigonometry: Advanced	106
34 Complex Numbers	109
35 Vectors and Matrices	111
36 Parameterized Equations	114
37 Polar Coordinates	116
38 Statistics	118
39 Limits	122

Tips on Math Homework

1.1 To solve any math problem, follow these four steps.

- (a) Understand the problem.
- (b) Devise a plan.
- (c) Carry out the plan.
- (d) Look back and check.

1.2 Follow the rules of acceptable mathematical writing.

- (a) Describe your approach at the beginning, if the solution is neither short nor simple.
- (b) Define variables unless no remote possibility of confusion.
- (c) Use mathematical notations correctly.
- (d) Treat mathematical expressions as nouns or sentences.
- (e) Follow the rules of grammar when combining words and expressions.
- (f) Make sure that your solution has a single flow.
- (g) State clearly your result in the final sentence.
- **1.3** Given a + b = 1, find the value of 2a + 2b. Two solutions are presented below. Only one is correct, even though both yield the correct answer.
 - I. Correct Solution

Because a + b = 1,

 $2a + 2b = 2(a + b) = 2 \times 1 = 2.$

II. Incorrect Solution

Because a + b = 1, assume a = 0.5 and b = 0.5. Then

 $2a + 2b = 2 \times 0.5 + 2 \times 0.5 = 2.$

Algebraic Expressions: Basic

2.1 Review basic formulas.

- (A) $(a+b)^2 = a^2 + 2ab + b^2$
- (B) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$
- (C) $(a-b)^2 = a^2 2ab + b^2$
- (D) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- (E) $(a-b)^3 = a^3 3a^2b + 3ab^2 b^3$
- (F) $a^2 b^2 = (a+b)(a-b)$
- (G) $a^3 b^3 = (a b)(a^2 + ab + b^2)$
- (H) $a^3 + b^3 = (a+b)(a^2 ab + b^2)$
- (I) $\left(a + \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} + 2$, where $a \neq 0$.
- **2.2** The middle number of three increasing consecutive odd numbers is n. Express the product of the three numbers in terms of n.
- **2.3** Four sides of a square are a units long. Express the area and perimeter of the square in terms of a.
- **2.4** The product of two numbers is 10. One of them is a. Express their sum in terms of a.
- **2.5** The first of three increasing consecutive even numbers is 2n 4. Express the last number in terms of n.
- **2.6** John drives from point A to B at speed of x miles per hour. On his way back, his speed is 10% faster. Which expression is his speed back?
 - (a) x + 0.10

[©] Qishen Huang

- (b) $x \times 0.10$
- (c) $x \times 1.10$

2.7 Sort the following values in ascending order:

$$0.5, -0.5, 0.5^2, \text{ and } \frac{1}{0.5}.$$

2.8 Value a satisfies -1 < a < 0. Sort these values in ascending order:

$$a, -a, a^2, \text{ and } \frac{1}{a}.$$

2.9 Find the values of x such that $x^2 = 0.36$.

- **2.10** Simplify expression $\sqrt{a^2b^2}$, where a < 0 < b.
- **2.11** Consolidate expression $10x^2 + [2x (5 + 4x^2 x) 3]$.
- **2.12** Consolidate expression $a^3 4 (a^2 5a) + (5a^2 3 6a^3)$.
- **2.13** Consolidate expression $(x+1)^2 + x(x-2y) 2x$.
- **2.14** Simplify expression $3ab^2 5a^2b + (-3a^2b) 4ab^2$.
- **2.15** Suppose a b + c d = a x. Express x in terms of a, b, c, and d.
- **2.16** Suppose $a^3 \times x \times a^{m+5} = a^{2m+8}$. Express x in terms of a and m.
- **2.17** Given equation (x-1)(x+7) = (x+1)(x-7) + y, express y in terms of x.
- **2.18** Suppose $z \div \left(\frac{3}{2}x^2y^2\right)^2 = 2x^3y^3$. Express z in terms of x and y.
- 2.19 Classify the following identities as true or false.
 - (A) $(-a^2)^3 = a^6$ (B) $a^3 + a^2 = a^5$ (C) $a^3 \times a^2 = a^6$ (D) $(a^3)^2 = a^6$ (E) $(3a)^3 = 9a^3$ (F) $a^6 \div a^3 = a^2$

2.20 Classify the following identities as true or false.

(a) $(x+1)^2 = 1 + 2x + x^2$

[©] Qishen Huang

(b) $(x-1)^2 = x^2 - 1$

(c)
$$x^3 + x^3 = x^6$$

2.21 Classify the following identities as true or false.

- (a) $(2xy) \cdot (-3xy) = -6xy$
- (b) $(x-y)(x+2y) = x^2 + xy 2y^2$
- (c) $(-4x^2)^3 = -12x^6$
- (d) $(x-y)^2 = (y-x)^2$

2.22 Identify the expressions that are always positive.

(A) a^2 (B) a + 2(C) |a + 1|(D) $a^2 + 1$ (E) $4 - (-a)^3$

2.23 Find the expressions that can have value of 0.

- (a) |x-1|
- (b) $x^2 + |y|$
- (c) $x^2 + |x 1|$
- **2.24** Does $a^4 > 0$ imply $a^5 > 0$?

2.25 Non-negative values a and b satisfy a + b = 0. Find the values of a and b.

2.26 Assume |x| = 3, |y| = 10, and xy < 0. Find all possible values of x - y.

2.27 Factor expressions in x.

(a)
$$x^2 - 4$$

(b) $x^4 - 64x^2$
(c) $x^3 - 2x^2 - 4x - 12$
(d) $(x - 1)(x - 2) - 6$

(e) $(x^2 - 2x)^2 - 2(x^2 - 2x) - 3$

2.28 Factor expressions in x and y.

(a)
$$4x^2 - 9y^2$$

- (b) $x^{2} 25 2xy + y^{2}$ (c) $xy^{2} + 4xy + 4x$ (d) $4x^{2} - y^{2} + 2x + y$ (e) $(x^{2} + y^{2})^{2} - 4x^{2}y^{2}$ (f) $x^{3}(x - y) + x^{2}(y - x)$
- **2.29** Multiply $(x y)(x^2 + xy + y^2)$.
- **2.30** Expand $(a + b c)^2$.
- **2.31** Multiply (2a + b)(2a b).
- **2.32** State the number of terms in expanded (a + b)(b + c).

2.33 Find Greatest Common Factors (GCFs) and Least Common Multipliers (LCMs).

- (a) x 1 and x + 1
- (b) $x^2 1$ and x + 1
- (c) $x^2 1$ and $x^3 + 1$

2.34 Find quotients and remainders.

- (A) $(x^3 + x^2) \div (x + 1)$ (B) $(x^6 + x^5 + x^4 + x^3 + x^2) \div (x^2 + 1)$ (C) $(ax^3 + 1) \div (x + 1)$
- (D) $(ax^3 + bx^2 + cx + d) \div (x 1)$

2.35 Complete squares in x in the expressions.

(a) $x^2 - 2x - 3$ (b) $-2x^2 - 8x - 9$

2.36 Complete squares in x and y in expression $x^2 - 2x - 3y^2 + 4y - 5$.

2.37 Solve equation |x + 1| = 0.

2.38 Given equation |x - y| + |y - z| = 0, identity true statements about the variables.

- (a) All variables are zero.
- (b) All variables are equal.
- (c) Exactly two variables are equal.
- (d) At least two variables are equal.

Algebraic Expressions: Intermediate

- **3.1** Equation $(-3a^mb^{2n-1})(3a^{1+n}b^m) = -9a^4b^4$ is true for all a and b and some unknown constants m and n. Find the values of m and n.
- **3.2** Find the minimum value of $1 + 3(3 x)^2$.
- **3.3** Find the expression that is always greater:

$$\frac{a^4 + 2a^2 + 4}{3}$$
 and $\frac{a^4 + a^2 + 1}{4}$.

- **3.4** Suppose a < -2. Simplify expression |2 |1 a||.
- 3.5 Find the minimum values of the two expressions.

(a)
$$|x-3|$$

(b) $|x-3| + |5-x|$

- **3.6** Compute $99999^2 + 199999$ without a calculator .
- **3.7** Compute $2008^2 64$ without a calculator.
- **3.8** Compute $777^3 776 \times 777 \times 778$ without a calculator.
- **3.9** Evaluate the value of $x^2 + 2x + 1$ at x = 9999 without a calculator.
- **3.10** If $3a^2 + 5b = 9$, compute the value of $1.5a^2 + 2.5b + 0.5$.
- **3.11** Solve equation $(x 78)^2 = (x 98)^2$.
- **3.12** Compute the sum of the roots of equation (x-1)(x+9)(x-5) = 0.

- **3.13** Values a and b satisfy $(a+1)^2 + (b-23)^4 = 0$. Compute the value of a^b .
- **3.14** Given a b = 1, compute the value of $a^3 3ab b^3$.
- **3.15** Given $x^2 + x 3 = 0$, compute the value of $x^4 + 2x^3 + x^2$ without solving the equation.
- **3.16** Suppose x + y = 6 and xy = 4. Find the value of $x^2y + xy^2$ without solving the equations.
- **3.17** Suppose a + b = 1 and $a^2 + b^2 = 2$. Find the value of $a^3 + b^3$ without solving the equations.
- **3.18** Define binary operation \otimes as follows:

$$a \otimes b = egin{cases} a^2b, & a \ge b; \ ab^2, & a < b. \end{cases}$$

Solve equation $3 \otimes x = 48$.

3.19 Suppose (x-a)(x+2) = (x+6)(x-b) is true for all $x \in R$. Find the values of a and b.

Rational Expressions

4.1 Solve for x in equation

$$\frac{1}{x} = \frac{2}{3} + \frac{3}{2}.$$

4.2 Simplify expression

$$\frac{(x+\Delta)^2 - x^2}{\Delta}.$$

4.3 Simplify expression

$$\frac{\frac{1}{x + \Delta} - \frac{1}{x}}{\Delta}$$

4.4 Find the values of A and B that satisfy

$$\frac{1}{x^2 - 6x + 5} = \frac{A}{x - 5} + \frac{B}{x - 1}.$$

(The right hand side is called partial fractions of the left hand side.)

4.5 List all possible values of $\frac{a}{|a|}$, where $a \neq 0$.

- **4.6** Suppose ab < 0. Compute all possible values of $\frac{a}{|a|} + \frac{b}{|b|}$.
- **4.7** Find all possible values of $\frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|}$, where $abc \neq 0$.
- **4.8** Suppose $b \neq 0$ and $\frac{a}{3} = \frac{b}{5}$. Compute the value of fraction $\frac{a}{b}$.

4.9 Suppose
$$\frac{a}{3} = \frac{b}{5} = \frac{c}{7}$$
, where $b \neq 0$. Compute the value of $\frac{a+b-2c}{b}$.
4.10 Simplify expression $\frac{(2a-2b)^6}{(b-a)^3}$, where $a \neq b$.
4.11 As x goes to infinity, what value does $\frac{2x-9999}{2x}$ approach?
4.12 Simplify expression $\frac{y}{5x-y} + \frac{5x}{y-5x} + 1$.
4.13 Simplify expression $\frac{x+x^2+x^3+x^4}{x^{-1}+x^{-2}+x^{-3}+x^{-4}}$.
4.14 Evaluate expression $\frac{x-3}{x^2-1} \div \frac{x^2-2x-3}{x^2+2x+1}$ at $x = \sqrt{5} + 1$.
4.15 Evaluate expression $x + \frac{x^2}{x-1}$ at $x = \sqrt{3} + 1$.

4.16 Suppose $xyz \neq 0$ and

$$\frac{\frac{1}{x}}{\frac{1}{y}} = \frac{x}{z}$$

Express z in terms of x and y.

4.17 Identify true identities.

(a)
$$x^2 \ge x$$
.
(b) $x \ge \frac{1}{x}$, where $x \ne 0$.
(c) $x^2 \ge 2x - 1$.

4.18 Simplify expression $\left(\frac{1}{x-y} + \frac{1}{x+y}\right) / \frac{xy}{x^2 - y^2}$.

4.19 If we increase x and y by 10%, by what percent does $\frac{x}{x+y}$ change?

4.20 Simplify expression
$$\frac{(-3a^3)^2}{a^2}$$
.

4.21 Simplify expression
$$(-2x)^2 + \frac{6x^3 - 12x^4}{3x^2}$$
.

4.22 Simplify expression $\frac{(x-y)^2 - (x+y)(x-y)}{2y}.$

 \bigodot Qishen Huang

4.23 Simplify expression $\frac{1-x^2}{x^3-3x^2+2x} + \frac{x+1}{x^2+x}$.

4.24 Observe

$$1 + \frac{1}{3} = \frac{2^2}{3}$$
 and $2 + \frac{1}{4} = \frac{3^2}{4}$.

Find the general pattern.

4.25 If $y \neq 0$ and 2x = 7y, compute the ratio of x : y.

4.26 If $\frac{x}{2} = \frac{y}{3}$, compute the value of $\frac{xy^2 + yx^2}{x^3 + y^3}$.

4.27 Given $y(x+y) \neq 0$ and $\frac{x}{x+y} = \frac{3}{5}$, compute the value of $\frac{x}{y}$.

4.28 Given a: b = 4: 7, identify true statements.

- (A) (a+1): (1+b) = 5:8
- (B) (b-a): b = 3:7

4.29 Evaluate the value of $\frac{x^2 - y^2}{x^2y + xy^2}$ at $x = \sqrt{5} + 1$ and $y = \sqrt{5} - 1$.

- **4.30** Polynomials f(x) and g(x) each have at least two unlike terms. Can f(x)g(x) and $\frac{f(x)}{g(x)}$ be a monomial?
- **4.31** Compute without a calculator:

$$\frac{123 \cdot 370 - 123}{122 \cdot 369 + 123}.$$

4.32 Given

$$\frac{1}{a - 100} = \frac{1}{b + 101} = \frac{1}{c - 102} = \frac{1}{d + 103}$$

sort a, b, c, and d in descending order.

4.33 Set $\{a, b, c\} = \{1234, 4567, 7890\}$. Choose the values of a, b, and c to minimize the value of

$$\frac{1}{a + \frac{1}{b + \frac{1}{c}}}$$

without computing any of its possible values.

4.34 Find the number of integer values of x that satisfy inequality

$$\frac{1}{3} > \frac{4}{x} > \frac{1}{5}.$$

4.35 Prove inequality $\frac{n}{n+1} < \frac{n+1}{n+2}$, where $n \ge 1$.

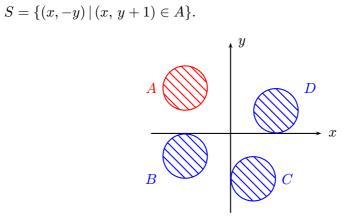
4.36 If $x^2 + y^2 = 2$ and $\frac{1}{x^2} + \frac{1}{y^2} = 1$, find the value of |xy| without solving the equations.

4.37 If $a + \frac{1}{a} = 5$, compute the value of $a^2 + \frac{1}{a^2}$.

Linear Relations: Basic

- 5.1 Express the set of points in the fourth quadrant.
- **5.2** Point (a, b) is in the 2nd quadrant. Find the quadrants the following points are in.
 - (A) (-a, b)
 - (B) (a-1, b)
 - (C) (a, b+1)
 - (D) (1-a, -b-2)

5.3 Given set A on the graph, identify the set S defined by



- **5.4** Line *l* is perpendicular to the *y* axis. Points P(1, 2) and *Q* are on line *l*. Find the *y* coordinate of *Q*.
- **5.5** Find where the lines x = 1 and y = -1 intersect, if at all.
- 5.6 Find the slope of a line connecting two given points in each case.

- (a) (0, 0) and (0, 1)
- (b) (0, 0) and (1, 0)
- (c) (0, 0) and (1, 1)
- (d) (-1, -1) and (1, 1)
- 5.7 Determine whether the three points are coline (on a single line):

A(1, 3), B(-2, 0), and C(2, 4).

- 5.8 Find the slopes of the following lines.
 - (A) A horizontal line.
 - (B) A vertical line.
 - (C) y = x + 1.
 - (D) y = -x + 1.
 - (E) 2x + 3y + c = 0, where c is some constant.
- **5.9** In each case, determine whether line AB is parallel or perpendicular to CD.
 - (a) A(0, 0), B(0, 1), C(0, 0), D(1, 0).
 - (b) A(0, 1), B(2, 3), C(4, 5), D(6, 7).
- **5.10** Given two points P(5, 1) and Q(8, 9), perform the following tasks.
 - (a) Write the equation of the line passing the two points.
 - (b) Express the set of all points on the line segment between P and Q.
 - (c) Express the set of all points on the line passing P and Q.
 - (d) Express the set of points on the line passing P and Q and in the first quadrant.
- **5.11** Find the coordinates of the midpoint M between two points P(0, 1) and Q(2, 3).
- **5.12** Find the two points that trisect the line segment between two points (1, 6) and (10, 21).
- **5.13** Given point P(1, 2), find point Q that is symmetric to P with respect to each of the following axes and points.
 - (a) The x axis.
 - (b) The y axis.
 - (c) The origin.
 - (d) Point W(5, 6).

- **5.14** Given points P(1, 6) and Q(8, 9), express the line segment symmetric to segment PQ with respect to each of the following axes and points.
 - (a) The x axis.
 - (b) The y axis.
 - (c) The origin.
 - (d) Point R(5, 6).
- 5.15 Determine the sign of the slope of the line that isn't any axis in each case.
 - (a) It is not present in the 3rd and 4th quadrants.
 - (b) It is not present in the 2nd and 4th quadrants.
 - (c) It is not present in the 2nd and 3rd quadrants.

5.16 Check whether the following points are on line y = 8x + 9.

- (a) (0, 1)
- (b) (0, 9)

5.17 How many points are needed to determine a line?

5.18 Are the two equations equivalent?

- (a) x + 2y = 9
- (b) 2x + 4y 18 = 0
- **5.19** Line l intercepts the x axis at 4 and the y axis at 5. Write its equation.

5.20 Write the two following equations in intercept form $\frac{x}{a} + \frac{y}{b} = 1$.

- (A) x + 2y = 3
- (B) 4x 5y = -6
- **5.21** Write an equation in slope-intercept form for the line with slope of 3 and y intercept of 2.
- **5.22** Write the following equations in slope-intercept form y = kx + b.
 - (A) x + 2y = 3
 - (B) 4x 5y = -6
- **5.23** Find the slope of linear equation $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are some non-zero constants.

- **5.24** Find the intercepts of linear equation y = kx + b, where $k \neq 0$.
- **5.25** Write an equation for the line l that passes point (1, 2) and is parallel to each line below.
 - (a) y = 2x + 1
 - (b) y = 6
 - (c) x = 8
- **5.26** Write an equation for the line l that passes point (1, 2) and is perpendicular to each line below.
 - (a) y = 2x + 1
 - (b) y = 6
 - (c) x = 8
- 5.27 Write equations of the lines passing the origin.
- 5.28 Compute the distance between the two points in each pair.
 - (a) (0, 1) and (2, 3)
 - (b) (1, 0) and (2, 0)
 - (c) (0, 3) and (0, 4)
 - (d) (-1, -1) and (1, 1)

5.29 Find the distance from point P(2, -3) to each axis.

- (a) The x axis.
- (b) The y axis.
- **5.30** Compute the area of the triangle formed by the x axis, the y axis, and line y = -x+2.
- **5.31** Find the distance between two parallel lines: x = 6 and x = -8.
- **5.32** Find the area of a triangle with vertices of A(0, 0), B(3, 4), and C(4, 3).
- **5.33** Find the value of k such that line y = kx + 3 passes point (1, 2).
- **5.34** John drives at constant speed of 40 miles per hour. Express the distance he travels as a function of time.
- **5.35** Solve equation |2x + 7| = 1.
- 5.36 Solve two equations in two variables.

[©] Qishen Huang

- (a) |x-1| + |x+y| = 0
- (b) $(x-1)^2 + y^2 = 0$

5.37 Positive integers a and b have no common factor except 1. In addition,

b = 5(a - b).

Find the values of a and b.

Linear Relations: Intermediate

- **6.1** Find line *l* that is half way between lines *m*: y = kx + a and *n*: y = kx + b, where $k \neq 0$ and $a \neq b$.
- **6.2** Find the distance between two parallel lines: y = 2x + 5 and y = 2x + 10.
- **6.3** Find the distance from point P(2, 3) to line l: x + 2y = 3.
- **6.4** Find the point Q on line l: x 2y = 1 that is closest to point P(1, 1).
- **6.5** Write an equation for the line *l* that is symmetric to line m: y = 2x + 1 with respect to each axis.
 - (a) The x axis.
 - (b) The y axis.
- **6.6** Show that line ax + by = c is symmetric to bx + ay = c with respect to line x y = 0.
- **6.7** Solve equation |2x + 8| = |4x 1|.
- **6.8** If $71_b = 5 \times 17_b$, find the value of base b.
- **6.9** Function f(x) = 6x + 8. Perform the following tasks.
 - (a) Find the inverse function of f.
 - (b) Determine whether the inverse function f^{-1} is linear.
 - (c) Compute the product of the slopes of functions f and f^{-1} .
 - (d) Show that f is symmetric to f^{-1} with respect to line y = x.
- **6.10** Functions f(x) = Ax + B and g(x) = Cx + D, where A, B, C, and D are constants and $AC \neq 0$. Identify linear functions on the following list.
 - (a) 2f(x) + 6g(x)

[©] Qishen Huang

- (b) f(x)g(x)
- (c) f(x) + 8
- (d) |f(x)|
- (e) f(g(x))
- (f) $2f^{-1}(x) + 6g(x)$

6.11 A linear function f has slope of 1. In addition, $f^{-1}(2) = 3$. Find the value of f(4).

- **6.12** Solve equation 5x 6y = 0, where x and y are positive integers less than 10.
- **6.13** Define function f(x) as follows:

$$f(x) = \begin{cases} x+1, & x \le 1; \\ 3-x, & x > 1. \end{cases}$$

Evaluate f(f(2)).

6.14 Given linear function f(x) = ax + b, define g(x) as follows:

g(x) = f(x+1) - f(x).

Write g in algebraic form.

6.15 Point (x_0, y_0) is not on line *l*: f(x, y) = 0. Define line *m* as follows:

$$f(x, y) - f(x_0, y_0) = 0.$$

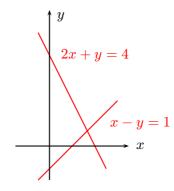
Answer two questions.

- (a) Is point (x_0, y_0) on line m?
- (b) Are the two lines parallel or perpendicular?
- **6.16** Write an equation of a plane with x intercept of 1, y intercept of 2, and z intercept of 3.

6.17 Define set S as

 $S = \{(x, y) \mid 2x + y \le 4, \ x - y \le 1, \ x \ge 0, \ y \ge 0\}.$

Mark the set on the graph.



Linear Relations: Advanced

7.1 Function f(x) = 3x + 4. Perform the following tasks.

- (a) Give the domain and range of the function.
- (b) Determine its monotonicity (increasing or decreasing).
- (c) Prove that for any x_1 and x_2 in the domain,

$$f(ax_1 + (1 - a)x_2) = af(x_1) + (1 - a)f(x_2)$$
, for any $a \in R$.

- **7.2** Function f(x) = ax + b. Point $C(x_C, 0)$ is the midpoint between points $A(x_A, 0)$ and $B(x_B, 0)$. Prove that $(x_C, f(x_C))$ is the midpoint between points $(x_A, f(x_A))$ and $(x_B, f(x_B))$.
- **7.3** Points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are above line l: y = ax + b. Prove all points in the set

$$S = \{ (kx_1 + (1-k)x_2, \, ky_1 + (1-k)y_2), \quad k \in (0, \, 1) \}$$

are above the line.

- **7.4** Show that the graph of equation $xy + y^2 = 0$ consists of two lines.
- 7.5 Graph the solution set of inequality

 $x^2y + xy^2 + xy \ge 0.$

- **7.6** Given points P(0, 0) and Q(2, 2), write an equation for all points A(x, y) such that |PA| = |QA|.
- **7.7** Solve equation |x |x 3|| = 5.
- **7.8** Solve equation |x |x |x 3||| = 5.

[©] Qishen Huang

- **7.9** Suppose x y + 0.5z = 10 and x + 3y + 1.5z = 90. Find the value of x + y + z.
- **7.10** Given an arbitrary equation f(x, y) = 0, find another equation that is symmetric to the equation with respect to line x + y 2 = 0.
- **7.11** Constants k_1 and k_2 satisfy $k_1k_2 = 1$. Find the axes of symmetry about which the line $y = k_1x$ is symmetric to $y = k_2x$.
- **7.12** Use techniques in linear equations to express $0.5\overline{168}$ as a fraction number.
- **7.13** Regardless of the value of parameter m, line (m + 3)x + (2m 1)y + 7 = 0 passes a fixed point. Find the point.

Word Problems: Basic

- 8.1 An 8 inch pizza is cut into 4 equal slices. A 10 inch pizza is cut into 6 equal slices. Which slices are larger?
- **8.2** A shirt is on sale at a 20% discount. The sales tax is 5%. A new sales clerk simply takes 15% off the original price. Is this correct? Why?
- 8.3 Twenty seven white unit cubes are stacked into one large cube. The surface of the large cube is then painted red. After the painting, how many small cubes are still completely white?
- **8.4** If the general inflation rate was 3% last year and John's salary grew 4%, how much did the purchasing power of his salary change?
- 8.5 Find the speed in degrees per minute of the minute hand of an accurate a clock.
- **8.6** At an office supplies store, pencils sell individually for \$0.10 each and in packs of 12 for \$0.80 per pack. Kay buys 2 packs instead of 24 individual pencils. How much money does she save?
- 8.7 The price of regular rice is \$1 per pound; that of premium rice is \$2. Mix 300 pounds of regular rice with 400 pounds of premium rice. What is the price of the mixture?
- **8.8** A car runs 30 miles per gallon of gasoline. How long can it run with \$40 worth of gasoline purchased at \$4 per gallon?
- 8.9 Mary and Jane shop at an office supplies store. Mary spends \$20 on 10 pens and 10 notebooks. Jane spends \$15 on 5 pens and 8 notebooks. Find the prices of the stationery.
- 8.10 A group of friends share cost of their party. If each pays \$42, there is surplus of \$19. If each pays \$40, the total is short of \$8. How much does the party cost?

[©] Qishen Huang

- 8.11 Dad is 42 years old and his son is 10. In how many years will Dad be 3 times as old as his son?
- 8.12 A sign on a freeway says 1 mile to the 15th Street exit and 4 miles to the Downtown exit. At a different location on the same road in the opposite direction, another sign says 2 miles to Downtown exit. How far is the sign from the 15th Street exit?
- 8.13 John is twice as old as her sister Jane. Four years ago, John was three times as old. What is John's current age?
- **8.14** The price of a stock is down 5% today. What percent increase tomorrow would return the stock to its original price?
- **8.15** A product is currently priced at \$10. If it is discounted by 10%, the profit margin will be 10%. What is the cost of the product?
- 8.16 A pair of shoes costs \$32 after a 20% discount. What is the original price?
- 8.17 A closing factory sold two used machines for \$1200 each. One machine brings 20% profit; the other 20% loss. How much is the total profit or loss?
- 8.18 Two toy stores, A and B, sell Game Boys. The regular price at store A is 80% of the manufacturer's suggested retail price. Store B normally does not discount at all. During Christmas season, store A discounts the product by 20% off its regular price. Store B discounts it by 40% off its regular price. Which store offers a better deal?
- **8.19** A store bought 10 vases for \$15 each from supplier A. It bought 40 vases for \$12.5 each from supplier B. Its profit margin is supposed to be 10%. How much should the selling price for each vase be?
- 8.20 Johnny has \$100 now and saves \$10 every month. His brother Jimmy has no money currently and is going to save \$15 per month. In how many months will Jimmy have as much money as Johnny?
- 8.21 A football team has won 10 games and lost 5. If the team wins the remaining games, they will have won 75% of all the games. How many more games will they play?
- 8.22 A baseball team won 20% of its games in the first half of the season. To achieve 50% winning average or better in the season, at least what percent of its remaining games must it win?
- 8.23 A pizza restaurant wants to make a pizza that is 10% bigger than the current 8 inch pizza. What is the diameter of the new pizza?
- **8.24** Four pizzas are ordered for the children at a party. Each pizza is cut into 8 equal slices. Each child eats 2 slices. There are 2 slices left. How many children are at the party?

[©] Qishen Huang

Word Problems: Intermediate

- **9.1** A mother horse and a baby horse walk 10 miles from point A to point B. The mother's speed is 10 miles per hour and the baby's 5 miles per hour. The mother rests for 2 minutes after every x minutes of walking. The baby keeps walking. They arrive together at the destination. Find the range of x.
- **9.2** A worker in a sunglass factory can make 50 frames or 100 lenses per day. There are 90 workers. How many workers should make lenses?
- **9.3** A store bought a number of tractors for \$800 each. Its regular selling price is \$1000. The store decides to discount the price to stimulate sales and to earn a minimum profit of 5%. What is the maximum discount rate?
- **9.4** Kate took a math test that she missed due to sickness. Her perfect score of 100 points raises class average from 80 points to 81. How many students including her in the class took the test?
- 9.5 John drives from point A to point B at a constant speed. On his way back, he drives 20% faster and spends 12 minutes less. How much time does he spend on the round trip?
- **9.6** An accurate clock shows exactly 3 pm. In how many minutes will the minute hand catch up with the hour hand?
- **9.7** A nut mix contains 10% of peanuts and 90% of cashews. It costs 5% more than pure peanuts. Are cashews more expensive than peanuts? By how much?
- **9.8** A mobile phone company offers two monthly calling plans, A and B. Under A, every minute costs \$0.10. Under B, the first 100 minutes costs \$12 and every extra minute costs \$0.08. What monthly talking time would make B the better choice?
- **9.9** It takes an escalator 20 seconds to move a person from the first floor to the second. If not operating, Jose takes 30 seconds to walk up on the still escalator. If Jose walks while the escalator is moving, how long does it take for him to reach the second floor?

- 9.10 A project must be finished in 80 days. Contractor A costs \$150 a day; Contractor B costs \$100 per day. The project manager has three possible choices:
 - (a) Hire contractor A alone. The project would be finished just in time.
 - (b) Hire contractor B alone. But it'd take 100 days.
 - (c) Mix use of contractor A and contractor B.

Find the solution to minimize total cost and to meet deadline.

- **9.11** A team of 4 people has just finished half of a project in 10 days. If the rest of the project needs to be done in 5 more days, how many more people are needed?
- **9.12** It takes Mary 6 days to complete a task. It takes Mike 8 days to do the same thing. If both work together, how long would it take?
- **9.13** Six people to can perform a task in 8 days. If we add 2 equally able people, how long would it take?
- **9.14** It takes 9 people 9 days to finish a project if they work 9 hours a day. If only 8 people work only 8 hours a day, how long would it take?
- **9.15** A 20 lb bag of nut mix has 40% of peanuts and 60% of cashews. To get 20% peanuts, how much of the mix should be replaced with pure cashews?
- **9.16** Bob walks from point A to point B at 6 miles per hour and back at 8 miles per hour. What is his average speed (total distance divided by total time)?
- **9.17** Tom drives 500 miles on the first day. On the second day, he drives twice as long and his average speed is 4/5 of that on the first day. How long does he drive on the second day?
- 9.18 On the eve of the Democratic Illinois primary, a poll finds 55% of the voters support candidate A, and 24% support B. Exactly 20% are undecided now and will support someone. Estimate the percentage of voters who will eventually support A.
- **9.19** Mary and Jane work part time for a library. Mary works every third day; Jane works every fifth day. Today, they are both working together. In how many days will they be working together again?

Word Problems: Advanced

- 10.1 Kay spends \$134 on 11 tickets of two different classes. Class A tickets are \$3 more expensive than class B. How many class A tickets and at what price does she buy?
- 10.2 Two classes took a test. Class A averaged 80 and class B averaged 90. The two classes together averaged about 83. Which class has more students?
- 10.3 In a family of five, members speak English, Spanish, or both. Two people speak Spanish, and four English. How many people speak both languages?
- 10.4 Students of a high school take a state assessment test. A class passes the test if half or more of the students pass. Which is greater: the percentage of passing classes or the percentage of passing students?
- 10.5 A Math quiz has 25 questions. A correct answer earns 8 points, and a wrong one costs 3 points. Jess scores 110 points. How many questions does Jess answer?
- 10.6 A man wants to see his girlfriend who lives 6 miles away. He leaves his house for her house at 6 am and walks at 4 miles an hour. When he leaves, he also sends a pigeon to her house. The bird flies at 30 miles an hour. When the bird reaches her house, she walks toward him at 2 miles an hour. When will they meet on the way?
- 10.7 An electric power company charges its consumers \$0.40 per kilowatt for the first 100 kilowatts in a month, \$0.50 per kilowatt for the second 100 kilowatts, and \$0.60 for each additional kilowatt beyond. Mary pays \$120 for electricity used in last month. How much electricity did her family consume in the month?
- 10.8 It takes Jose 12 days to complete a task. The same task would take Hans 24 days. After the two men work on the task for 4 days, Jose leaves for vacation and Hans continues. In how many more days will Hans finish the work?

10.9 Sergei makes a round trip from home to the library. The distance from home d (in meters) is a function of time t (in minutes):

$$d(t) = \begin{cases} 50 t, & 0 \le t \le 20; \\ 1000, & 20 < t < 30; \\ 2500 - 50 t, & 30 \le t \le 50. \end{cases}$$

How far is the library from his home? How long does he stay in the library? How fast does he walk home?

- 10.10 Jim and John drive from point A to point B in separate cars. Jim leaves at 6 am and arrives at 4 pm. John leaves at 10 am and arrives at 3 pm. Assume both men drive at constant speeds. Find when John catches up with Jim.
- 10.11 John and Bill walk towards to each other's house. If both leave at 10 am, they meet 10 minutes later. If Bill leave 3 minutes later then John, they'll meet 9 minutes after Bill leaves. How long does it take for Bill to walk to John's house?
- 10.12 Wang starts a project, which he can finish in 20 days. Five days later after he starts, his company sends Lee to help. Together, they finishes the project in 5 more days. If Wang is paid \$100 a day, what is a fair daily salary for Lee?
- 10.13 Todd agreed to work for his Dad for 20 days to get \$500 and an iPod. He worked for only 10 days and had to quit for some reason. He got \$200 and the iPod based on the amount of work done. How much was iPod valued?
- 10.14 A clock is broken. But its hour hand, minute hand, and second hand still move at constant speeds in the normal direction. The second hand passes the hour hand every 10 seconds (measured by an accurate clock) and passes the minute hand every 20 seconds. How often does the minute hand pass the hour hand?
- 10.15 In a group of tennis players, a person plays either single or mixed double, but not both. 1/2 of the men and 1/3 of the women play mixed double. What fraction of the group play mixed double?

Geometry: Basic

11.1 Answer the questions regarding points and lines.

- (a) Point P is not on line l. How do you find a point on the line so that the two points have the shortest distance?
- (b) Point P is not on line l. How many lines passing P are parallel to l?
- (c) Point P may and may not be on line l. How many lines passing P are perpendicular to l?

11.2 Identify true statements.

- (a) A point in geometry has no size.
- (b) A line in geometry has no length, direction, or thickness.
- (c) Two congruent triangles are similar.
- (d) Corresponding altitudes of two congruent triangles are equal.
- **11.3** Identify true statements.
 - (a) The ratio of a pair of corresponding altitudes of two similar triangles is equal to the ratio of any pair of corresponding sides.
 - (b) The ratio of the areas of two similar triangles is equal to the ratio of a pair of corresponding sides.
 - (c) All isosceles right triangles are similar.
 - (d) All equilateral triangles are similar.

11.4 Identify true statements about $\triangle ABC$.

- (1) If $\angle A > \angle B$, then a > b.
- (2) If a > b, then the altitude on side a is shorter than that on b.

[©] Qishen Huang

11.5 Two quadrilaterals ABCD and $A_1B_1C_1D_1$ satisfy

 $\angle A = \angle A_1, \ \angle B = \angle B_1, \ \angle C = \angle C_1, \text{ and } \angle D = \angle D_1.$

Are the quadrilaterals similar?

11.6 Two quadrilaterals ABCD and $A_1B_1C_1D_1$ satisfy

$$AB = A_1B_1$$
, $BC = B_1C_1$, $CD = C_1D_1$, and $DA = D_1A_1$.

Are the quadrilaterals congruent?

- 11.7 Answer the questions regarding points, tangent lines, and circles.
 - (a) Point P is outside circle C. How many tangent lines of the circle pass the point?
 - (b) Point P is outside circle C. How would you find the point on the circle that is closest to P?
 - (c) Two circles of different radii do not intersect. How would you find one point on each circle such that the two points have the longest distance among similar points?
 - (d) A line and a circle do not intersect. How would you find one point on the line and another on circle such that the two points have the shortest distance among similar points?
- 11.8 Find the measure of the angle that the second hand of an accurate clock rotates in 20 seconds.
- 11.9 State the sum of all angles of a triangle and that of a quadrilateral.
- 11.10 Find the polygons below that are symmetric with respect to some point.
 - (a) Parallelogram
 - (b) Equilateral triangle
 - (c) Rhombus
 - (d) Trapezoid
- 11.11 Find the number of axes of symmetry of the following shapes.
 - (a) Equilateral triangle
 - (b) Square
 - (c) Rhombus but not a square
 - (d) Regular hexagon
 - (e) Circle

[©] Qishen Huang

- **11.12** Given $\triangle ABC$ with each condition below, determine whether it is a right triangle in each case.
 - (1) The ratio of three angles is 1:2:3.
 - (2) The ratio of three sides is 3:4:5.
 - (3) $a^2 = (c+b)(c-b).$
- 11.13 A square and a circle have equal areas. Which shape has shorter perimeter?
- 11.14 Compute the area of a triangle with three sides of 6, 8, and 10.
- **11.15** A right triangle has an acute angle of 30° and a hypotenuse of 1 unit long. Find the lengths of the two other sides.
- **11.16** Two sides of a right triangle are 1 unit long. Find the length of the third side.
- 11.17 The three sides of an equilateral triangle are 1 unit long. Find the altitudes of the triangle.
- **11.18** What is circumcenter of a triangle? Is it inside the triangle? In $\triangle ABC$, the perpendicular bisectors of sides AB and BC intersect at point D. Connect D with the midpoint E of side AC. Find the measure of $\angle AED$.
- 11.19 What is centroid of a triangle? In $\triangle ABC$, lines AD and BE each divide the triangle into two equal areas. The two lines intersect at point F. Connect points C with F and extend the line segment to divide the triangle into two areas. Find the ratio of the two areas.
- **11.20** What is the orthocenter of a triangle? Is it inside the triangle? $\triangle ABC$ has area of 10. Lines AD and BE are altitudes on sides BC and AC respectively. AD and BE intersect at point F. Connect points C with F and extend the line segment to intersect side AB at point G. Compute product $CG \times AB$.
- 11.21 What is the incenter of a triangle? Is the center equally distant from the three sides or three vertices of the triangle?
- **11.22** Find how many diagonals can be drawn from a vertex of convex *n*-sided polygon (n > 3).
- 11.23 State the number of diagonals in a quadrilateral and that in a hexagon.
- **11.24** Find the sum of all angles in a *n*-sided polygon $(n \ge 3)$.
- 11.25 Every interior angle of a huge regular polygon is 172°. How many sides does the polygon have?
- 11.26 If an interior angle of a polygon is 45° , what is the corresponding exterior angle?

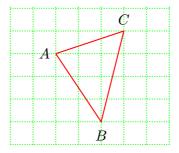
[©] Qishen Huang

- 11.27 Compute the sum of all exterior angles of a polygon.
- 11.28 At least how many acute angles does a triangle have? At most how many acute angles does a quadrilateral have?
- **11.29** If all interior angles of a polygon are equal, are all the sides equal?
- **11.30** In $\triangle ABC$ and $\triangle A_1B_1C_1$,

 $AB = 2, BC = 3, AC = 4, A_1B_1 = 4, B_1C_1 = 6, \text{ and } A_1C_1 = 8.$

The altitude on side AB is CD, and that on side A_1B_1 is C_1D_1 . Find the ratio of $CD: C_1D_1$.

- 11.31 If the base of a triangle increases by 5% and its corresponding altitude decreases by 5%, by what percent does the area of the triangle change?
- 11.32 Lengthening the three sides of a triangle by 100%, we get a new larger triangle. By what percents do its three angles, perimeter, and area change?
- **11.33** The shadow of a 1-meter tall tree is 0.4 meter long. If the shadow of another tree is 2 meters, how tall is the tree?
- **11.34** On the graph, each grid is 1×1 . Compute the area of $\triangle ABC$.



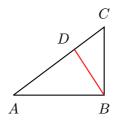
- 11.35 The central angle of a sector of a unit circle is 50° . Find the area of the sector.
- 11.36 A sector of a unit circle has perimeter of 2.5. Find the area of the sector.
- 11.37 Two circles do not intersect and are outside each other. How many tangent lines can they have in common?
- 11.38 Two circles of radii 1 and 2 intersect at only one point. How many tangent lines can they have in common?
- **11.39** A unit circle is inscribed in $\triangle ABC$. In the triangle, $\angle A = 60^{\circ}$. Connect the circle center O with vertex A. Find the length of segment AO.

[©] Qishen Huang

- 11.40 Point A is outside a circle. The longest distance between the point and the circle is 9 and the shortest 5. Find the radius of the circle.
- **11.41** The two diagonals of a parallelogram intersect at point A. Find the minimum angle the shape rotates around A to coincide with the original shape.
- 11.42 Prove that the area of a rhombus is half of the product of its two diagonals.
- 11.43 Draw two diagonals of a parallelogram (not a rhombus or rectangle). Find the number of pairs of congruent triangles in the parallelogram.
- 11.44 An angle of an isosceles triangle is twice as big as another. Find the measure of the smallest angle of the triangle.
- 11.45 Three sides of a right triangle are 3, 4, and 5. Find the altitude on the hypotenuse.
- 11.46 Every side of a regular hexagon is 1. Find the distance from the center to any side.
- **11.47** The sides of $\triangle ABC$ are a, b, and c. Determine the sign of expression

 $a^2 - 2ab + b^2 - c^2$.

- **11.48** $\angle A$ of $\triangle ABC$ is greater than the corresponding exterior angle. Is the triangle acute, right, or obtuse?
- **11.49** In right $\triangle ABC$, BD is the altitude on side AC. Find the number of pairs of similar triangles in $\triangle ABC$.



- **11.50** If the two diagonals of a quadrilateral are perpendicular to each other, is the polygon a rhombus?
- 11.51 In $\triangle ABC$, points D and E are the midpoints of sides AB and AC respectively. In addition, BC = 10. Find the length of segment DE.
- 11.52 The three midsegments of a triangle form another smaller triangle. The perimeter of the smaller triangle is 10. Find the perimeter of the original triangle.

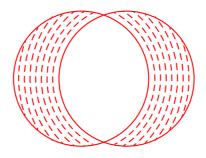
Geometry: Intermediate

- 12.1 Three distinct points are on a plane. Draw a line through every pair of two points. How many distinct lines are obtained?
- **12.2** One hundred lines, $l_1, l_2, l_3, ..., l_{100}$, satisfy

 $l_1 \perp l_2, \ l_2 \perp l_3, \ \cdots, \ l_{99} \perp l_{100}.$

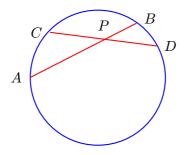
Is line l_1 parallel or perpendicular to l_{100} ?

12.3 A couple of unit circles overlap partially. Find the perimeter of the non-overlapping region.

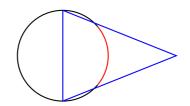


12.4 Two chords AB and CD of a circle intersect at point P inside the circle. Prove that

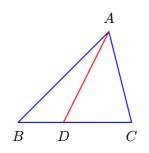
 $AP \cdot PB = CP \cdot PD.$



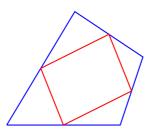
12.5 Three angles of an isosceles triangle are 70°, 70°, and 40°. The base of the triangle is a diameter of a circle. Find the measure of the minor arc of the circle that the two equal sides intercept.



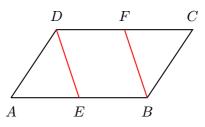
12.6 The area of $\triangle ABC$ is 25. Point *D* is on side *BC* such that BD = 4 and DC = 6. Find the area of $\triangle ABD$.



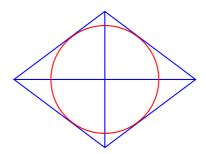
12.7 Connecting clockwise the midpoints of the four sides of a quadrilateral, we get a smaller quadrilateral inside. Prove the new quadrilateral is a parallelogram.



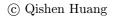
12.8 In parallelogram ABCD, point E on side AB and point F on side CD satisfy AE = CF. Connect point D with E, and B with F. Prove that quadrilateral BFDE is a parallelogram.

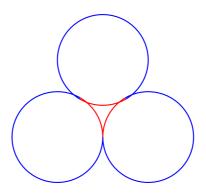


12.9 Each side of a rhombus is 5. One of its diagonals is 8. Find the radius of the inscribed circle.

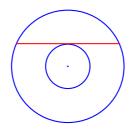


- **12.10** An equilateral triangle has an inscribed circle and a circumscribed circle. Find the ratio of their radii.
- 12.11 Three unit circles are mutually tangent. Find the perimeter of the gap area that the circles enclose.

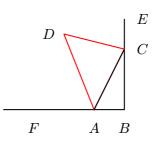




12.12 The radii of two concentric circles are 5 and 2. A chord of the bigger circle is tangent to the small circle. Find the length of the chord.



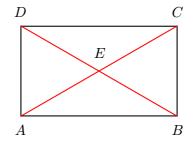
- 12.13 A quadrilateral has an inscribed circle. In addition, the sum of the top side and the bottom is 8. Find the perimeter of the quadrilateral.
- **12.14** In $\triangle ABC$, $\angle B = 80^{\circ}$. The bisectors of the two exterior angles at A and C intersect at point D. Find the measure of $\angle ADC$.



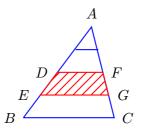
12.15 The perimeter of a triangle is an odd integer. In addition, its two sides are 2 and 9. Find all possible values of the third side.

[©] Qishen Huang

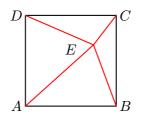
- 12.16 Two diagonals of a rhombus are equal. Is it a square?
- **12.17** The two diagonals of a rectangle ABCD intersect at point E. In addition, $\angle AEB = 120^{\circ}$. Find the measure of $\angle ADE$.



- **12.18** The sides of $\triangle ABC$ are a, b, and c. Show that |a b| < c.
- **12.19** The diagonals of a quadrilateral are two diameters of its circumcircle. What is the shape of the quadrilateral?
- **12.20** Two equilateral triangles have sides of 0.5 and 3 units long. How many of the smaller triangles are needed to fill the bigger triangle without overlapping?
- **12.21** $\triangle ABC$ has area of 16. Three lines parallel to side BC divide side AC into four equal segments. As a result, three non-overlapping trapezoids are produced. Find the area of the middle trapezoid.



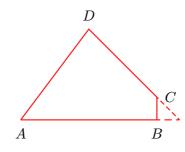
12.22 Point E is a point inside unit square ABCD. Connecting E with the four vertices, we have four triangles. Find the sum of the areas of the top and the bottom triangle.



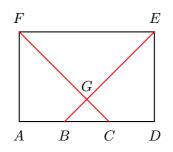
Geometry: Advanced

13.1 In a convex polygon, at most how many interior angles are acute?

13.2 In quadrilateral ABCD, $\angle A = 60^{\circ}$ and $\angle D = 70^{\circ}$. Opposite sides AB and CD do not intersect. Find the measure of the angle formed by the their extensions.

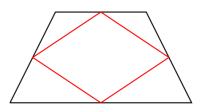


13.3 In rectangle ADEF, AF = 4 and AD = 6. The bisectors of top two angles $\angle E$ and $\angle F$ and the bottom side AD form a triangle. Find the area of the triangle.

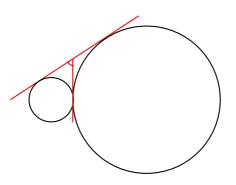


13.4 Each altitude of four congruent quadrilaterals is longer than 1. Try to cover a unit circle completely with the four such quadrilaterals without overlapping.

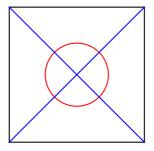
13.5 Connecting clockwise the midpoints of adjacent sides of an isosceles trapezoid, we get a new smaller quadrilateral. What kind of quadrilateral is it?



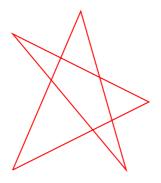
- **13.6** Two intersecting circles have radii of 1 and 3. Find the range of possible distance between their centers.
- **13.7** A unit circle is inscribed in a trapezoid. Find the minimum of the perimeter of the trapezoid.
- 13.8 Two circles of radii 1 and 3 are outside each other and mutually tangent. Draw a common tangent line through the tangent point and another common tangent line. Find the measure of the angle included by the two lines.



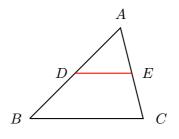
13.9 A circle passes all four points that trisect the two diagonals of a unit square. Find the diameter of the circle.



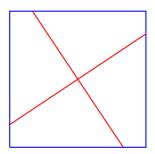
13.10 A star has five vertices. Find the sum of the angles at those vertices.



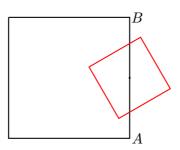
- **13.11** Given a rectangle and a point, find a line passing the point and dividing the rectangle into two regions of equal areas.
- 13.12 A square is just big enough to contains a unit circle. What is radius of the largest circle in one of the corners in the square but outside the unit circle?
- **13.13** $\triangle ABC$ has area of 1. Its side BC is 2. Line DE is parallel BC. If $\triangle ADE$ has area of 0.5, find the length of DE.



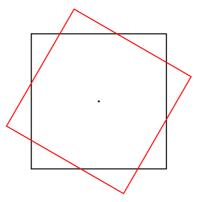
- 13.14 A circle is inscribed in a unit square, and a smaller square is inscribed in the circle. Find the area of the smaller square.
- 13.15 Two mutually perpendicular lines pass the center of a unit square and divide the square into four regions. Prove that the four regions have equal areas.



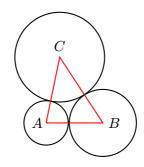
- 13.16 Find the altitude of the smallest cone in terms of volume that contains a unit sphere.
- **13.17** The center of a 5×5 square is at midpoint of one side of a 8×8 square. Find the size of the overlapping area.



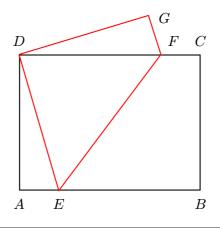
- **13.18** We cut a circle sector of 120° from a unit circle. The remaining is used to form a cone. Find the radius of the base circle and the height of the cone.
- 13.19 Rotate a unit square clockwise around its center by 30°. Find the overlapping area of the new and the old square.



- **13.20** The two shorter sides of a right triangle are a and c. Find the radius of its inscribed circle.
- **13.21** Connect the centers of three mutually tangent circles of radii 2, 3, and 4 to form a triangle. Find the area of the triangle.



13.22 Fold a 6×8 rectangle so that two opposite vertices coincide. Find the overlapping area.



Radicals

14.1 Simplify expression $\sqrt[3]{x\sqrt[3]{x\sqrt[3]{x}}}$.

14.2 Find true identities and inequalities.

- (a) $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$, where $a \ge 0$ and $b \ge 0$.
- (b) $\sqrt{a-b} = \sqrt{a} \sqrt{b}$, where $a \ge 0$ and $b \ge 0$.
- (c) $\sqrt{ab} = \sqrt{a}\sqrt{b}$, where $a \in R$.
- (d) $\sqrt{a^n} = (\sqrt{a})^n$, where $a \ge 0$ and $n \in \mathbb{Z}$.
- (e) $\sqrt{a} \leq a$, where $a \geq 0$.
- (f) $\sqrt{a} \ge 0$, where $a \ge 0$.
- (g) $\sqrt{a^2} = a$, where $a \in R$.

14.3 Find true identities and inequalities.

- (a) $\sqrt[3]{ab} = \sqrt[3]{a}\sqrt[3]{b}$, where $a \in R$.
- (b) $\sqrt[3]{a/b} = \sqrt[3]{a}/\sqrt[3]{b}$, where $b \neq 0$.
- (c) $\sqrt[3]{a^n} = (\sqrt[3]{a})^n$, where $a \in R$.
- (d) $\sqrt[3]{a} \leq a$, where $a \in R$.
- (e) $\sqrt[3]{a} \ge 0$, where $a \in R$.
- (f) $a\sqrt[3]{a} \ge 0$, where $a \in R$.

14.4 Simplify expressions without a calculator.

(a)
$$\sqrt{18} - \sqrt{8}$$

(b) $\sqrt{5} - \sqrt{45}$
(c) $\sqrt{(\sqrt{2} + \sqrt{5})^2}$

- (d) $\sqrt{(-3)^2}$
- (e) $(1+\sqrt{5})(\sqrt{5}-2)$
- (f) $(\sqrt{5} \sqrt{3})(\sqrt{5} + \sqrt{3})$
- (g) $(5+\sqrt{6})(5\sqrt{2}-2\sqrt{3})$
- (h) $\sqrt{7 + 2\sqrt{10}}$

14.5 Suppose that 0 < a < 1 and n is a positive integer. Identity true inequalities.

- (a) $\sqrt[n]{a} \ge 0$
- (b) $\sqrt[n]{a} < 1$
- (c) $\sqrt[n]{a} > a$

14.6 Find the range of x that makes $\sqrt{3-x}$ a real number.

- 14.7 Find the value of x that makes $\sqrt{1-x^2}$ the largest.
- **14.8** If n is an odd positive number, can \sqrt{n} be possibly an even number?
- **14.9** Find integer n such that $n \leq \sqrt{98} < 1 + n$.
- 14.10 Compare the values without a calculator.

(a)
$$\sqrt{10} - \sqrt{9}$$
 versus $\sqrt{11} - \sqrt{10}$.

(b) $\sqrt{n} - \sqrt{n-1}$ versus $\sqrt{n+1} - \sqrt{n}$, where $n \ge 1$.

14.11 Compare the values without a calculator.

(a)
$$\sqrt{4 \cdot 6}$$
 versus $\frac{4+6}{2}$.
(b) $\sqrt{5 \cdot 5}$ versus $\frac{5+5}{2}$.
(c) \sqrt{ab} versus $\frac{a+b}{2}$, where $a \ge 0$ and $b \ge 0$.

14.12 Compare the values without a calculator.

(a) 3√2 versus 2√3
(b) √√3 versus ³√3
(c) √2 versus ³√3

14.13 Rationalize the denominators.

(a)
$$\frac{1}{2+\sqrt{3}}$$

(b)
$$\frac{1}{\sqrt{3} - \sqrt{2}}$$

(c)
$$\frac{1}{\sqrt{n+2} + \sqrt{n}}$$

14.14 Identify true statements below.

(a) $\sqrt{2} + \sqrt{3} = \sqrt{5}$ (b) $5\sqrt{3} - 2\sqrt{3} = 3\sqrt{3}$ (c) $\sqrt{5} - \sqrt{3} = \sqrt{2}$ (d) $5 + \sqrt{5} = 5\sqrt{5}$

14.15 Rationalize the denominator in expression $\frac{\Delta}{\sqrt{x+\Delta}-\sqrt{x}}$.

14.16 Consolidate expression $\frac{\sqrt{x}}{1+\sqrt{x}} + \frac{1-\sqrt{x}}{\sqrt{x}}$ into a single fraction.

14.17 Use formula $(\sqrt{s} \pm \sqrt{t})^2 = s \pm 2\sqrt{st} + t$ to simplify expressions below.

(a) $\sqrt{11 - 2\sqrt{30}}$ (b) $\sqrt{19 - 8\sqrt{3}}$

 ${\bf 14.18} \ {\rm Simplify \ expression}$

$$\frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \dots + \frac{1}{\sqrt{8} + \sqrt{9}}$$

14.19 How many positive values of x make $\sqrt{10000 - x}$ an integer?

Exponentials: Basic

15.1 Review formulas.

$a^{xy} = (a^x)^y$	$a^{x+y} = a^x a^y$
$a^{x-y} = \frac{a^x}{a^y}$	$e^{\ln a} = a$
$\log(ab) = \log a + \log b$	$\log(\frac{a}{b}) = \log a - \log b$
$\log a^b = b \log a$	$\log a^{\frac{1}{b}} = \frac{1}{b} \log a$
$\log_a b = \frac{\log b}{\log a}$	$\log_{\frac{1}{a}} b = -\log_a b$

15.2 Recognize common mistakes. The identities are generally false.

$$a^{x\pm y} = a^x \pm a^y \qquad a^{xy} = a^x \cdot a^y$$
$$a^{\frac{x}{y}} = \frac{a^x}{a^y} \qquad \log(a \pm b) = \log a \pm \log b$$
$$\log(ab) = \log a \cdot \log b \qquad \log \frac{a}{b} = \frac{\log a}{\log b}$$
$$\log(a^b) = (\log a)^b$$

- **15.3** We know that $\log(a^b) = (\log a)^b$ is generally false. Can you find an instance that the equality holds?
- **15.4** Assume a > 0, $a \neq 1$, b > 0, and $b \neq 1$. Prove the two identities.

(A)
$$\log_a b = \frac{1}{\log_b a}$$

(B)
$$\log_{\frac{1}{a}} b = \log_a \frac{1}{b}$$

15.5 Let $f(x) = \log x$, where x > 0. Find true identities.

- (a) f(xy) = f(x)f(y), where x > 0 and y > 0.
- (b) f(xy) = f(x) + f(y), where x > 0 and y > 0.
- (c) $f(x^y) = (f(x))^y$, where x > 0.
- (d) $f(x^y) = yf(x)$, where x > 0.

15.6 Let $f(x) = e^x$. Find true identities.

- (a) f(xy) = f(x)f(y)(b) $f(xy) = (f(x))^y$ (c) f(x+y) = f(x) + f(y)(d) f(x+y) = f(x)f(y)
- 15.7 Evaluate the following expressions without a calculator.
 - (a) $\log_3 1$
 - (b) $\log_3 3$
 - (c) $\log_3 \sqrt{3}$
 - (d) $\log_{\sqrt{3}} 3$
 - (e) $\log_{\frac{1}{\sqrt{3}}} 3$
 - (f) $\log_3 9$
 - (g) $\log_3 \frac{1}{3}$
 - (h) $\log_3(\log_3(\log_3 27))$

15.8 Evaluate the following expressions without a calculator.

(a)
$$\log_3 2 + \log_3 \frac{1}{2}$$

(b) $\frac{1}{2} \log 2 + \frac{1}{2} \log 5$
(c) $\frac{\log_3 4}{\log_3 2}$
(d) $\log_5 4 \cdot \log_4 5$
(e) $\log_8 9 \cdot \log_3 2$
(f) $\log_4 8 \cdot \log_8 16 \cdot \log_{16} 64$

15.9 Evaluate the following expressions without a calculator.

- (a) $\log_2 5 \cdot (\log_5 2 + \log_{25} 0.5)$
- (b) $\log 20 + \log_{100} 25$
- (c) $\log_4 9 + \log_2 \frac{4}{3}$
- (d) $(\log_5 7)(\log_7 9)(\log_9 11)\cdots(\log_{23} 25)$
- **15.10** Given $s = 1 + \pi^t$, express $1 + \pi^{-2t}$ in terms of s.
- **15.11** Simplify expression $3^{-(4a+1)} 3^{-(4a-1)} + 3^{-4a}$.
- **15.12** Given $\log_5 15 = a$, express $\log_3 45$ in terms of a.
- **15.13** Given a > 0, b > 0, and $\log_{a^3} b^2 = c \neq 0$, express $\log_{b^4}(a^5)$ in terms of c.

15.14 Simplify expressions.

(a)
$$\log_{\frac{1}{2}} \frac{1}{3}$$

(b) $\log(3^{2^2})$
(c) $(\sqrt{3})^{\log_3 6}$
(d) $\left(\frac{1}{2}\right)^{-1-\log_2 9}$
(e) $\log_5 11 \cdot \log_9 5 \cdot \log_{11} 9$
(f) $\frac{1}{\log_2 24} + \frac{1}{\log_3 24} + \frac{1}{\log_4 24}$

15.15 Solve equations.

- (a) $\log_x 5 = 5$
- (b) $\log_3 x = -2$
- (c) $\log_{3x} 3 = 3$
- (d) $\log_2 x = \log_x 2$
- (e) $\log_2(x^2) = 4$

15.16 Solve equations and one inequality.

- (a) $2^{x} = 3$ (b) $2^{3x} = 3^{2x}$ (c) $3^{3x} \cdot 4^{4x} = 5^{5x}$
- (d) $x^{\log x} = 10$

(e) $x^{2x-1} = x^{x+4}$

(f)
$$\left(\frac{1}{3}\right)^x - 4 > 0$$

15.17 Consider the graph of function $y = \log x$. Perform two tasks.

- (a) Find asymptotes, if any.
- (b) Find intercepts, if any.
- **15.18** Consider the graph of function $y = e^x$. Perform two tasks.
 - (a) Find asymptotes, if any.
 - (b) Find intercepts, if any.
- **15.19** Define functions $f(x) = \log x$ where x > 0, and $g(x) = e^x$ where $x \in R$. Find the domains of composite functions f(g(x)) and g(f(x)).
- **15.20** If $a \in (0, 1)$ and $b \in (0, a)$, is $\log_a 3 > \log_b 3$?
- **15.21** If $a \in (0, \infty)$, $b \in (0, a)$, is $\log_{\frac{1}{2}} a > \log_{\frac{1}{2}} b$?
- 15.22 Are the two functions increasing or decreasing?
 - (a) $f(x) = \log_3 x$, where x > 0.
 - (b) $g(x) = \log_{\frac{1}{2}} x$, where x > 0.
- **15.23** Function $y = \log x$ is increasing on its domain $(0, \infty)$. Graph the function on a calculator. Does it increase faster near x = 8 or near x = 2?
- **15.24** Function $y = e^x$ is increasing on its domain $(-\infty, \infty)$. Graph the function on a calculator. Does it increase faster or x = 4 than near x = 2?
- **15.25** Functions f(x) and g(x) are both increasing. In addition, the range of g is in domain of f. Prove that composite function f(g(x)) is increasing.
- 15.26 Find domains of the functions and then simplify the functions.
 - (a) $f(x) = \ln(e^x)$ (b) $g(x) = \sqrt[3]{e^{\ln(x^3)}}$
- **15.27** Find whether function $f(x) = -\log_{\frac{1}{2}}(-x)$ is increasing or decreasing on interval $(-\infty, 0)$.
- **15.28** Find the monotonic (decreasing or increasing) intervals and range of function $f(x) = \log(4-x)$.

[©] Qishen Huang

- **15.29** Find the monotonicity of function $f(x) = 2^{\frac{1}{x+1}}$ on $(-\infty, -1)$.
- **15.30** Find the inverse of function $f(x) = 3 \log_2(5x 2)$, where $x > \frac{2}{5}$.
- 15.31 A fixed-rate saving account carries annual interest rate of 5%. The owner deposits \$100 into the account. In how many years will the balance become \$200? Use a calculator if necessary.

Exponentials: Intermediate

16.1 Given $4^a = 10$, find the value of 8^{-2a} without a calculator.

- **16.2** Solve equation $\log_x(\sqrt{3}-1) = -1$.
- **16.3** Solve equation $\log_2 \log_3 \log_4 x = 1$.

16.4 Compute

$$\log_{10}\frac{1}{2} + \log_{10}\frac{2}{3} + \log_{10}\frac{3}{4} + \dots + \log_{10}\frac{9}{10}.$$

16.5 Find the inverse of function $f(x) = 8e^{3x-2} - 1$.

16.6 Find the monotonic interval(s) and range of function $f(x) = e^{x^2 - 4x + 2}$.

16.7 Find the domain, monotonic intervals, intercepts, asymptotes, and inverse of function

$$f(x) = \frac{1}{1 - e^{-x}}$$

- **16.8** Find the monotonic intervals and range of function $f(x) = \log_3(8 + 2x x^2)$.
- **16.9** Find the domain of function $f(x) = 2 \log \frac{10 x}{10 + x}$. Determine whether the function is even or odd.

16.10 Find the value(s) of x such that $\frac{18}{7+e^x}$ is an integer.

- **16.11** Find the integer n that minimizes $|3^n 8^5|$. Use a calculator if necessary. Do not use it for more than two values of n.
- 16.12 Without solving the following equation, find the product of its roots.

 $\log^2 x + 2 \cdot \log x + \log 2 \cdot \log 3 = 0.$

[©] Qishen Huang

16.13 A positive and geometric sequence $\{a_n, n = 1, 2, 3, ...\}$ has common ratio of r which is greater than 1. Is sequence $\{\log_{\frac{1}{2}} a_n, n = 1, 2, 3, ...\}$ arithmetic or geometric, increasing or decreasing?

Exponentials: Advanced

17.1 Simplify expressions.

- (a) $8^{\frac{2}{3}} (\log_3 18 \log_3 2)^2$
- (b) $\log 500 + \log \frac{8}{5} \frac{1}{2} \log 64$ (c) $\log_4 \left(\sqrt{2 + \sqrt{3}} + \sqrt{2 - \sqrt{3}} \right)$
- **17.2** Suppose that $10^a = 20$ and $20^b = 10$. Find the value of product *ab* without a calculator.
- 17.3 Solve inequality

$$\frac{2\log_x 3 - 1}{3 - \log_x 3} > 1, \text{ where } x > 0.$$

- 17.4 Solve equation $5^x + 5^{x+1} + 5^{x+2} + 5^{x+3} = 312$.
- **17.5** Solve equation $e^x e^{-x} = e$.
- **17.6** Solve equation $5^{x+1} 3^{x^2-1} = 0$.
- **17.7** Solve equation $9^x = 27^{3x}$.
- **17.8** If $9^{2x} 9^{2x-1} = 18$, find the value of 3^x .
- **17.9** Solve equation $9^x 6 \cdot 3^x 7 = 0$.
- **17.10** Solve equation $4^x + 6^x 12 \cdot 9^x = 0$.
- **17.11** Find the inequalities that are true for all $a \in R$.
 - (A) $\log_{\frac{1}{2}}(a^2+1) > \log_{\frac{1}{2}}a^2$

[©] Qishen Huang

(B) $2^{a^2+1} \ge 2^{2a}$

17.12 Solve equation $\log_3 \log_9 x + \log_9 \log_3 x = 1$.

17.13 Solve inequality $-1 < \log_{\frac{1}{x}} 10 < -\frac{1}{2}$.

- **17.14** Solve equation $\log_2(x-1) = 2 \log_2(x+1)$.
- **17.15** Solve inequality $\log_2(4+3x-x^2) \log_2(2x-1) > 1$.
- **17.16** Solve inequality $\log_{\frac{1}{8}}(2-x) > -\log_{8}\frac{1}{x}$.
- **17.17** Find whether function $y = 3^{|x|}$ is increasing on $(-\infty, \infty)$.

17.18 Find the increasing interval of function $f(x) = \left(\frac{1}{3}\right)^{x^2 - 2x + 3}$.

- **17.19** The graph of function $f(x) = a^x + b$ passes point (1, 3). In addition, $f^{-1}(2) = 0$. Find the values of a and b.
- **17.20** Point $(2, \frac{\sqrt{2}}{2})$ is on graph of function $f(x) = x^a$. Find the value of a.
- **17.21** Find a positive value a such that the domain and the range of function $f(x) = a^x 1$ are both [0, 2].
- **17.22** Function y = f(x) where $x \in (0, \infty)$ has the property:

f(st) = f(s) + f(t) for any s and t.

Perform the following three tasks.

- (a) Find the value of f(1).
- (b) Prove that $f(\frac{1}{x}) = -f(x)$.
- (c) Give an example of such function.
- **17.23** Function f has domain of R. In addition, $f(\ln x) = x + 1$ for all $x \in (0, \infty)$. Find the form of f(x).
- **17.24** Find the point on line x + 2y = 1 that minimizes the value of $2^x + 4^y$.

17.25 Find the minimal value of function $f(x) = \frac{e^x + e^{-x}}{2}$.

- **17.26** Find where function $f(x) = 3^{2x} 3^{x+2}$ reaches its minimum value.
- **17.27** Find the minimum of function $f(x) = 4^{x-\frac{1}{2}} 3 \cdot 2^x$ on [2, 4].

[©] Qishen Huang

17.28 Define function f as follows:

$$f(x) = \begin{cases} 2e^{x-1}, & x < 2;\\ \log_3(x^2 - 1) - 1, & x \ge 2. \end{cases}$$

Solve inequality f(x) > 1.

17.29 Find the range of function $f(x) = \frac{1}{2^x - 1}$.

17.30 Solve equation

$$\left(\sqrt{5-\sqrt{24}}\right)^x + 3\left(\sqrt{5+\sqrt{24}}\right)^x = 4.$$

General Functions

- **18.1** Give a non-constant function on R with no increasing and no decreasing interval.
- **18.2** The domain of function f(x) is (0, 2). Find the domain of $g(x) = f(x^2)$.
- **18.3** Which groups have two identical functions?
 - (a) f(x) = x and $g(x) = (\sqrt{x})^2$
 - (b) $f(x) = \frac{x^2}{x}$ and $g(x) = \sqrt{x^2}$
 - (c) f(x) = x and $g(x) = \log 10^x$
- **18.4** The solution set of equation f(x) = 0 is set A, that of g(x) = 0 is set B. Find the solution set for equation f(x)g(x) = 0. Is $A \cap B$ the solution set for f(x) + g(x) = 0?
- **18.5** Functions f and g have domain of R. In addition, the minimums of f and g are 2 and 3, respectively. Identify true statements.
 - (a) The minimum of f(x) + g(x) is 5.
 - (b) The minimum of f(g(x)) is f(3).
 - (c) The minimum of f(g(x)) is 2.
 - (d) $f(x)g(x) \ge 6$, for all $x \in R$.
- **18.6** Function $f(x) = ax^3 + bx 1$, where a and b are constants. In addition, f(2) = 3. Find the value of f(-2).
- **18.7** A function f(x) on $(0, \infty)$ satisfies f(st) = f(s) + f(t), where s and t are any values in its domain. Find the value of f(1).
- **18.8** Some non-constant function f(x) where $x \in R$ satisfies f(s+y) = f(s)f(y), where s and y are any values in its domain. Can you find a specific point on its graph?

18.9 Function f(x) passes point (0, 1). Which point below is on function g(x) = f(x+4)?

- (a) (4, -1)
- (b) (-4, 1)
- (c) (1, -4)
- (d) (1, 4)
- **18.10** Function f(x) is defined on R. a and b are two different values in R. What is the geometric interpretation of expression

$$\frac{f(a) - f(b)}{a - b}?$$

- **18.11** Function f(x) is defined on R. Determine whether function g(x) = f(x) f(-x) is odd or even.
- **18.12** Suppose f(x) is an arbitrary function on R. Is function $g(x) = f(x^2)$ odd or even?
- **18.13** Odd functions f(x) and g(x) share the same domain. Is function h(x) = f(x)g(x) odd, even, or neither?
- **18.14** Is the graph of $f(x) = 3^x$ symmetric to that of $g(x) = -3^{-x}$ with respect to the x axis, y axis, line y = x, or point (0, 0)?
- **18.15** Suppose that f(x) is an even function and g(x) an odd function. Both domains are $x \neq \pm \pi$. In addition,

$$f(x) + g(x) = \frac{1}{x - \pi}$$
, where $x \neq \pm \pi$.

Find analytic forms of f(x) and g(x).

- **18.16** Function $f(x) = |x^3 + 1| + |x^3 1|$. If $a \in R$, which points below are on the graph of the function?
 - (1) (a, -f(a))
 - (2) (a, f(-a))
 - (3) (-a, -f(a))
 - (4) (-a, -f(-a))
- **18.17** Suppose f(x) is an even function on R and it is decreasing on $(0, +\infty)$. Compare the values of f(1.4), f(1.5), and $f(-\sqrt{2})$.
- **18.18** The domain of an odd function f(x) is R. When $x \ge 0$, $f(x) = -x^2 + 2x$. Find analytic form of f(x) when x < 0.

[©] Qishen Huang

- **18.19** Function f satisfies 2f(x) f(2-x) = x + 2 for all x in its domain R. Find analytic form of f(x).
- **18.20** Define f(x) as the sum of the biggest digit and the smallest in integer x. For examples,

f(123) = 3 + 1 = 2 and f(2) = 2 + 2 = 4.

Define $f^{2}(x) = f(f(x))$, and so on. Find the value of $f^{168}(1234567890)$.

- **18.21** Function f is defined on all positive integers. f(n) is the remainder when n is divided by 10. Which statements are true?
 - (a) f(10n) = 0, where $n \ge 1$.
 - (b) f(n+9) = f(n-1), where n > 1.
 - (c) If m and n are in the domain and m > n, then f(m) > f(n).
- **18.22** Define [x] as the largest integer not more than x. Solve equation x[x] = 18.

18.23 Find graphs of the following functions.

(a) y = 2 - x(b) $y = \sqrt{1 + x^2}$ (c) $y = x^2 + x$ (d) $y = x^3 - x^2 + x$ (e) y = 1 - |x|(3) (2) (1) (4) (5)/

Inverse Functions

19.1 What is the inverse of the inverse of function f?

- **19.2** The inverse function of f(x) is $f^{-1}(x)$. Which statements are true?
 - (a) The domain of f is the range of f^{-1} .
 - (b) The range of f is the domain of f^{-1} .
 - (c) The domain of function $h(x) = f(f^{-1}(x))$ is that of f(x).
- **19.3** Invertible function f(x) is defined on R. Find the value of $f^{-1}(f(1))$.
- **19.4** Find the point that is symmetric to point A(1, 2) about line y = x.
- **19.5** Find the line that is symmetric to line y = 2x + 1 with respect to line y = x.
- 19.6 Identity true statements below.
 - (a) An increasing function is invertible.
 - (b) An invertible function on R is either decreasing or increasing on the whole domain.
 - (c) If a function has same value at two different points, it is not invertible.
- **19.7** Determine whether these functions are invertible. Find inverses of the invertible functions.
 - (a) f(x) = x 1.
 - (b) $f(x) = x^2$, where $x \in R$.
 - (c) $f(x) = x^2$, where $x \in (1, 2)$.
 - (d) $f(x) = x^3$.

- **19.8** Find the inverse function of $f(x) = 2 + 2^{-x}$, where $x \in R$. Specify the domain of the inverse function, if not R.
- **19.9** The inverse functions of f(x) and g(x) are f^{-1} and g^{-1} respectively. The range of g is in the domain of f. Show that function h(x) = f(g(x)) is invertible. Express its inverse function in terms of f^{-1} and g^{-1} .
- **19.10** The domain and range of increasing function f are both R. Prove that f^{-1} is increasing.
- **19.11** Suppose the domain and range of invertible function f are both R. Constant b is not zero. Express the inverse functions in terms of f^{-1} .
 - (A) g(x) = f(x) + b
 - (B) g(x) = bf(x)
 - (C) $g(x) = (f(x))^3$
- **19.12** Linear functions f(x) = 2x + m and g(x) = nx + 3 are mutually inverse functions. Find the values of m and n.

19.13 Find the inverse function of $f(x) = \log \frac{x+1}{x-1}$, where x > 1.

- **19.14** Find the inverse function of $f(x) = 2^x \frac{1}{2^x}$.
- **19.15** Define mapping $f: (x, y) \longrightarrow (x + 2y, 3x 4y)$. Find $f^{-1}(1, -2)$.

Quadratic Functions: Basic

20.1 Write the quadratic equations in vertex form (standard form): $y = a(x - h)^2 + k$.

- (1) $y = x^2 + 2x + 1$
- (2) $y = -x^2 + 2x + 1$
- (3) $y = 2x^2 + 3x$
- (4) $y = 5x^2 + 6x + 7$
- (5) y = 2(x+3)(x-4)

20.2 For each parabola below, find the vertex, opening direction, and axis of symmetry.

- (a) $y = x^2 + 2x + 1$
- (b) $y = -2(x-1)^2 + 2$
- (c) y = 2(x-1)(x+2)
- **20.3** Parabola $y = x^2 + bx + c$ is symmetric with respect to line x = 5. Find the value of b.
- **20.4** Consider parabola $y = a(x h)^2 + k$, where h and k are some constants. State the necessary and sufficient condition for each property below.
 - (1) It opens up.
 - (2) It does not intersect the x axis.
 - (3) It intersects the x axis at only one point.

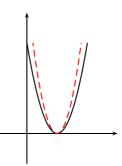
20.5 Identity true statements about quadratic function $y = ax^2 + bx + c$, where $a \neq 0$.

- (1) It reaches either maximum or minimum at the vertex.
- (2) Its domain consists of an increasing and a decreasing interval.
- (3) Its whole graph can be in a single quadrant.

- (4) It has only one axis of symmetry.
- (5) It crosses the y axis at only one point.
- **20.6** Suppose that a, b, and c are some constants and $a \neq 0$. Compare the shape and vertex of each parabola below to those of $y = ax^2 + bx + c$.
 - (1) $y = ax^2 + (b+1)x + c$

(2)
$$y = ax^2 + bx + (c+1)$$

- (3) $y = -ax^2 + bx + c$
- **20.7** The two curves represent two parabolas: $y = x^2 + bx + c$ and $y = 1.2x^2 + b_1x + c_1$, where a, b, b_1 , and c_1 are some constants. Which curve represents $y = x^2 + bx + c$?



20.8 A quadratic function y = f(x) satisfies f(4) = f(5). Find its axis of symmetry.

20.9 Write a quadratic function that satisfies each condition.

- (A) It passes three points (0, 4), (1, 9), and (-1, 3).
- (B) Its vertex is at (2, 1). It passes point (1, 2).

20.10 Write the new equation after each operation on parabola $y = 3x^2 + 4x + 5$.

- (a) Shift it horizontally by -2 units.
- (b) Shift it vertically by 2 units.

20.11 Write the new equation after each operation on parabola $y = 3x^2 + 4x + 5$.

- (a) Flip it along the x axis.
- (b) Flip it along the y axis.
- (c) Flip it along line x = 1.

20.12 How does one get parabola $y = ax^2 + bx + c$ from $y = ax^2$?

[©] Qishen Huang

20.13 Write the new equation after each operation on parabola $y = 3(x-4)^2 + 5$.

- (a) Rotate it clockwise around the vertex by 90° .
- (b) Rotate it clockwise around the origin by 90° .
- 20.14 How many possible points can two different quadratic functions intersect at? Why?20.15 Find the monotonic intervals and range of each quadratic function.
 - (A) $y = (x h)^2 + k$
 - (B) $y = -x^2 + bx + c$
 - (C) f(x) = -(x+5)(x-3)

20.16 Function $f(x) = x^2 - 4x + 3$. Find its maximum and minimum on each interval:

- (a) [1, 3]
- (b) [3, 4]
- 20.17 Given any three points, is there always a parabola that passes them?
- **20.18** Suppose quadratic equation f(x) = 0 has two different real roots, r_1 and r_2 . Is the identity true for some constant a:

$$f(x) = a(x - r_1)(x - r_2)$$
?

- **20.19** Solve equation $2x^2-5x-7=0$ by using the quadratic formula. Then write $2x^2-5x-7$ in form of $a(x-r_1)(x-r_2)$.
- **20.20** Compute the discriminants of the next two quadratic equations. Compare the result in (A) to that in (B). What do you observe from the comparison?
 - (A) $2x^2 + 3x + 1 = 0$ (B) $2x^2 - 3x + 1 = 0$
- **20.21** Find whether the equation $x^2 + \sqrt{2}x + 1 = 0$ has any real solutions.
- **20.22** If a < 0, does inequality $ax^2 + bx + c < 0$ always have a real solution?
- 20.23 Solve quadratic equations.
 - (a) $2x^2 4x 3 = 0$
 - (b) (x-1)(x+2) = 4
 - (c) $(x+1)^2 = 4x$
 - (d) $(3x-1)^2 = 25$

- (e) $(3x-1)^2 = 2(3x-1)$
- (f) $(3x-4)^2 = (4x-3)^2$
- (g) $x^2 + (x+1)^2 = (x+2)^2$

20.24 First guess what the solutions of the inequalities are like. Then solve the inequalities.

- (a) (3x+1)(x-2) < 0
- (b) $x^2 + x + 1 > 0$

Quadratic Functions: Intermediate

- **21.1** Solve equation $x^2 15|x| 100 = 0$.
- **21.2** Solve inequality $x^2 + 3|x| < 10$.
- **21.3** Solve inequality $|x^2 4| 3x \ge 0$.
- **21.4** Solve inequality $|x^2 4x + 4| > |x^2 3x 4|$.
- **21.5** Quadratic function y = f(x) is symmetric with respect to line x = 3. In addition, f(4) = 0. Find another zero of f(x).
- **21.6** Find the sum and product of the two roots in quadratic equation $2x^2 + 13x 31 = 0$.
- **21.7** Equation $x^2 3x + m = 0$ has a root of -1. Find the value of m.
- **21.8** Point P(0.5, 0.5) is on line x + y = 1, find two points on the line that are 1 unit from P.
- **21.9** Find the maximum area of a rectangle that is inside the triangle formed by the two axes and line y = 2 x.
- **21.10** Function $s = 600 t 4t^2$ is the distance of a landing aircraft runs on the runway before a full stop, where t is time in seconds on the runway. How much time doe it take for the plane to stop?
- **21.11** Function $f(x) = x^2$, where $x \in [0, +\infty)$. As x increases, y increases. Does y increase faster near x = 2 than near x = 1? Use a graphing calculator if necessary.
- **21.12** Function $f(x) = ax^2 + bx + c$, where a, b, and c are some constants. Define functions g and h as follows:

$$g(x) = f(x+1) - f(x),$$

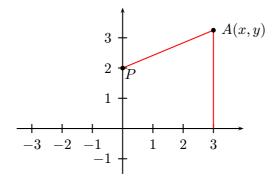
 $h(x) = g(x+1) - g(x).$

[©] Qishen Huang

Find algebraic form of h(x).

21.13 Point P(0, 2) is not on line l: y = 0. A(x, y) is a moving point.

- (a) Express the distance between point A and P.
- (b) Express the distance between point A and line l.
- (c) Write an equation for the trajectory of all A whose distance to P is equal to its distance to line l. What type of equation is this?



- **21.14** A product was discounted twice by the same percentage. The original price was \$100 and the current price is \$81. Find the discount percentage.
- **21.15** Solve the following higher degree equations.
 - (a) $x^8 + x^4 = 2$ (b) $x^4 - 20x^{-4} = 1$
- **21.16** Solve equation $(2x)^3 = (3x)^2$.
- **21.17** Solve equation $\frac{1}{x^2} \frac{2}{x} 3 = 0.$
- **21.18** Solve equation $e^{2x} + 2e^x 3 = 0$.
- **21.19** Solve equation $(\ln x)^2 2\ln x 3 = 0$.

Quadratic Functions: Advanced

- **22.1** Parabola $y = x^2 + bx b$ passes a fixed point regardless of the value of b. Find the point.
- **22.2** The vertex of $y = x^2 + 2x + c$ is on the x axis. Find the value of c.
- **22.3** Point (a, b) is in the third quadrant. In which quadrant is the vertex of parabola $y = ax^2 + bx$?
- **22.4** Parabola $y = ax^2 + bx + c$ is in 1st, 3rd, and 4th quadrant but not the 2nd quadrant. Which quadrant is its vertex in? Does the parabola open up or down?
- **22.5** Do graphs of $y = x^2$ and y = x 1 intersect?
- **22.6** A quadratic equation has only rational coefficients. If one root is irrational, is the other irrational or rational?
- **22.7** Find all real roots of equation $x^4 + 2x^2 24 = 0$.
- **22.8** The solutions of equation $33x^2 + 99x 9999 = 0$ are x_1 and x_2 . Compute the value of $(x_1 1)(x_2 1)$ without solving the equation.
- **22.9** Write a quadratic function with only rational coefficients and a zero of $1 + \sqrt{3}$.
- **22.10** The two sides of a right triangle are 2 and 3. Find all possible values of the third side.
- 22.11 The product of two consecutive odd integers are 143. Find the smaller integer.
- **22.12** Find the minimum value of $x^2 + y^2$ for all points (x, y) on line 2x + y = 1.
- **22.13** Function $f(x) = ax^2 + bx + c$. Its two zeros r_1 and r_2 satisfy $1 < r_1 < 2 < r_2$. Find the sign of product f(1)f(2).

22.14 Find the parabola symmetric to the following parabola with respect to the origin:

$$y = ax^2 + bx + c.$$

22.15 Find a function that is symmetric to the following function with respect to line y = x:

 $y = (x - 1)^2$, where $x \ge 1$.

22.16 A quadratic equation $x^2 + bx + c = 0$ has two roots x_1 and x_2 . Find another quadratic equation with roots of $2x_1$ and $2x_2$. Express the new equation in terms of b and c.

Polynomial and Rational Functions

23.1 Is a linear function a polynomial function?

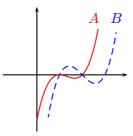
- **23.2** Find a factor of $x^7 + a^7$.
- **23.3** Polynomial function f(x) has degree of 3. Define function g as follows:

g(x) = f(x) - f(x - 1).

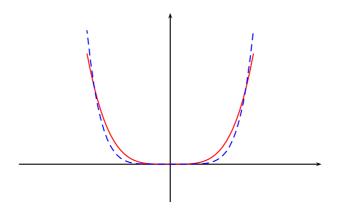
Show that g is a polynomial function. What degree does g(x) have?

- **23.4** Consider polynomial function f(x) = (x+1)(x-2)(x-3).
 - (a) Find all zeros.
 - (b) The zeroes divide R into a few open intervals. Find sign of f(x) for x in each interval.

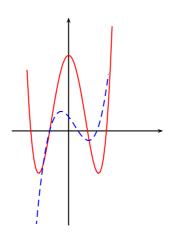
23.5 Polynomial function f of degree 3 satisfies f(1) = f(2) = 0. Which curve can it be?



23.6 The two curves represents two functions $f(x) = x^6$ and $g(x) = x^4$. Which curve is f?



23.7 Polynomial functions f and g have degrees of 3 and 4 respectively. Which curve below is f(x)?



- **23.8** A polynomial of degree 7 has all its distinct zeros in interval [1, 5] and f(-1) = 1. What is the sign of f(6)?
- **23.9** The constant term in expanded polynomial $(x + 3)^n$ is 81. Find the value of integer n.
- **23.10** Compute the sum of all coefficients including the constant term in expanded $(1+x^7)^7$.

23.11 Find the constant term in expanded expression $\left(2x - \frac{1}{\sqrt{x}}\right)^6$.

- **23.12** Find the sum of the coefficients of terms $x^1, x^3, x^5, \ldots, x^9$ in expanded $(1-x)^9$.
- **23.13** Suppose f(x) is a polynomial function. Prove that if f(c) = 0 for some c, then x c is a factor of f(x).
- **23.14** Find the remainder when $x^7 x^3$ is divided by x 2.

[©] Qishen Huang

23.15 Provide a polynomial function f(x) of degree 3 satisfying

$$24 - f(5) = f(1) = f(6) = f(8) = 0.$$

23.16 Show that 7 is the only real solution of equation $x(x+1)(x+2) = 7 \times 8 \times 9$.

23.17 It is obvious that 1 is a zero of the following two functions. Find its multiplicity.

- (a) $x^4 1$
- (b) $(x-1)^4$

23.18 The zeros of polynomial function q(x) are 1, 2, and 3. p(x) is another polynomial function. Find the domain of rational function $\frac{p(x)}{q(x)}$.

23.19 Find the domain and asymptotes of function $f(x) = \frac{2}{x-1}$.

23.20 Find the asymptotes of function $f(x) = \frac{x^2 + 2}{x+1}$.

- **23.21** Identify true statements.
 - (a) A rational function always has a vertical asymptote.
 - (b) If a rational function doesn't have a horizontal asymptote, it is not bounded at both ends.

23.22 Find the minimum value of $y = x + \frac{1}{x}$, where x > 0.

23.23 Solve equation

$$1 - \frac{1}{\frac{1}{1 + \frac{1}{2 - x}}} = 3.$$

23.24 Solve rational equation $\frac{2}{x+2} = \frac{4}{x^2-4}$. 23.25 Solve rational equation $\frac{1}{x-2} = \frac{3}{x}$. 23.26 Solve rational equation $\frac{x-1}{x-3} = \frac{-2}{3-x}$. 23.27 Solve rational equation $\frac{x}{x-1} - \frac{2}{x+1} = \frac{4}{x^2-1}$.

23.28 Solve rational inequality $\frac{x+1}{x-1} > 0$. **23.29** Solve inequality $\frac{x^2 - 6x + 5}{x - 1} > 0.$ **23.30** Solve inequality $\frac{x^2 - 6x + 5}{x^2 - 2x - 3} > 1$. **23.31** Solve inequality $\frac{x-6}{x-3} > \frac{x-1}{x-2}$. **23.32** Solve inequality $\frac{1}{|2x-1|} \ge \frac{1}{x}$. **23.33** Solve inequality $\frac{1}{2|x|-1} \ge \frac{1}{x}$.

23.34 How would one obtain the graph of $y = \frac{1}{1+x}$ from that of $y = \frac{1}{x}$? Solve rational

$$\begin{cases} \frac{6}{x} + \frac{6}{y} = 1, \\ \frac{14}{x} + \frac{4}{y} = 1. \end{cases}$$

23.36 A rectangle is formed by the two axes and a vertex on curve $y = \frac{2}{x}$, x > 0. Find its area.

23.37 Two positive values x and y satisfy x + y = 1. Find the minimum of $\frac{1}{x} + \frac{1}{y}$.

Radical Equations and Functions

24.1 Solve equation $\sqrt{99} = 99\sqrt{x}$.

24.2 Find the domain of function $f(x) = \frac{\sqrt{2-x}}{x-1}$. **24.3** Find the domain and range of function $y = \sqrt{-x^2 - 6x - 5}$. **24.4** Solve equation $|x - 27| + \sqrt{x - 28} = x$. **24.5** Solve equation $\sqrt{3x} + \sqrt{8x} = \sqrt{x}$. **24.6** Solve inequality $(\sqrt{5} - \pi)(\sqrt{x} - \pi) < 0$. **24.7** Solve equation $\sqrt{2x+1} = \frac{1}{2}\sqrt{3}$. **24.8** Solve equation $\sqrt{x+5} - \sqrt{x+3} = 1$. **24.9** Solve equation $\sqrt{3-x} + \sqrt{3+x} = x$. **24.10** Solve equation $\sqrt{x} = x - 2$. **24.11** Solve equation $\sqrt{x} = x - 2$. **24.12** Solve equation $\frac{|x| - 3}{2x^2 - x - 15} = 0$. **24.13** Solve equation $\sqrt[3]{x+19} - \sqrt[3]{x} = 1$. **24.14** Solve inequality $\sqrt{2-x} < x$. **24.15** Solve inequality $\sqrt{x^2 - 3x + 2} > x - 3$. **24.16** Solve equation $\frac{\sqrt{x+2} + \sqrt{x-2}}{\sqrt{x+2} - \sqrt{x-2}} = 3$.

24.17 Solve equation $\sqrt{x} + \sqrt{x+2} - \sqrt{2x+3} = 0.$

24.18 Solve equation system involving two variables

$$\sqrt[3]{x+1} + \sqrt[3]{y-1} = 2,$$

 $x+y = 26.$

Circles

- **25.1** Express the set of the points above the x axis and outside a unit circle centered at the origin.
- 25.2 Identify the subsets among the three sets:

$$\begin{split} C &= \{(x,y): \, x^2 + y^2 \leq 1\},\\ S &= \{(x,y): \, |x| \leq 1, \, |y| \leq 1\}\\ D &= \{(x,y): \, |x| + |y| \leq 1\}. \end{split}$$

25.3 Find the radius and center of each circle below.

(a) $x^2 + y^2 = 1$ (b) $x^2 + y^2 - 2x = 3$ (c) $x^2 + y^2 - 4x + 6y = 6$

25.4 Find the radius and center of circle $x^2 + ax + y^2 + by + c = 0$.

25.5 How many points are necessary to determine a circle?

25.6 Rotate the circle clockwise around its center by 30° :

$$(x-1)^2 + (y-2)^2 = 3^2$$

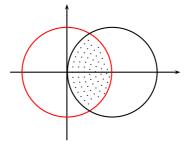
Write the equation of the new circle.

25.7 How can one get graph of $(x-1)^2 + (y-2)^2 = 9$ from $x^2 + y^2 = 9$?

25.8 Write equations for the circles satisfying the conditions.

(a) The radius is 3 and the center is at the origin.

- (b) The radius is 1. Lines x = 6 and y = 8 each divide the circle into two half circles.
- (c) Point A(1, 2) is on the circumference. Among all other points on the circumference, point B(3, 4) is the farthest from A.
- **25.9** Point (1, 2) is on a circle centered at the origin. Find three other points on the circle with rational coordinates.
- **25.10** Write the equation of a circle that passes three points: A(0, 0), B(0, 3), and C(4, 0).
- **25.11** Write the equation of the circle inscribed in the square with three vertices: O(0, 0), A(0, 4), and B(4, 0).
- **25.12** Rotate circle $(x 1)^2 + (y 1)^2 = 4$ clockwise by 45° around the origin. Write the equation of the new circle.
- **25.13** A circle centers at the origin and intersects with the two axes at four points. The shortest distance between two intersection points is 2. Write the equation for the circle.
- **25.14** Write the equation of a circle with center at (0, 1) and a tangent line x + y = 4.
- **25.15** Given circle $x^2 + y^2 = 1$, write the equations of the tangent lines with slope of 2.
- **25.16** Given circle $x^2 + y^2 = 1$, find the equation of the tangent line that passes point $P(1, \sqrt{3})$.
- **25.17** Find the intersection points of two circles, $x^2 + y^2 = 1$ and $x^2 + y^2 + 2x 2y 3 = 0$.
- **25.18** Two tangent lines of circle $x^2 + y^2 = 1$ pass point P(2, 0). Find the equations of the lines.
- **25.19** Identify shape represented by equation $(x^2 + y^2 + 1)(x^2 + y^2 3) = 5$.
- **25.20** Circle A is symmetric to circle B: $x^2 + y^2 = 1$ with respect to line *l*: x y 1.2 = 0. Write the equation of circle A.
- **25.21** Find the area of the portion of $x^2 + (y+1)^2 = 4$ above the x axis.
- **25.22** Find the area of the overlapping region of two circles $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$.



© Qishen Huang

- 25.23 Find the maximum area of a rectangle inside a unit circle centered at the origin.
- **25.24** A circle is in the 1st, 2nd, and 3rd quadrants, but not the 4th. Which quadrant is its center in?
- 25.25 Find the radii of the circles satisfying the two requirements:
 - (a) The circles are in the second quadrant and tangent to both axes.
 - (b) Point (-4, 4) is on their circumferences.
- **25.26** Find the point on upper half of circle $x^2 + y^2 = 1$ that maximizes $f(x, y) = (x y)^2$.
- **25.27** Find equation of the smallest circle that contains two circles: $x^2 + y^2 = 1$ and $(x 1)^2 + (y + 1)^2 = 1$.
- 25.28 How do we intersect a cone with a plane to get the following figures?
 - (a) A circle.
 - (b) A line.
 - (c) A point.

Ellipses

- **26.1** Is a circle is a special ellipse?
- 26.2 How do we intersect a cone with a plane to get an ellipse?
- **26.3** Write ellipse $x^2 + 2y^2 + 6x 4y = 1$ in standard form.
- **26.4** We stretch a unit circle centered at the origin horizontally in both left and right directions by 100%. Write the new equation. What shape is it?
- 26.5 Identify shapes.

(a)
$$x^{2} + 2y^{2} = 3$$

(b) $x^{3} + 2xy^{2} = 3x$
(c) $x^{2} - 4y^{2} = 5$

(c)
$$x^2 - 4y^2 = 5$$

(d) $x^2 + 6y^2 = 7$

(d)
$$x^2 + 6y^2 = -7$$

(e) $x^2 + 8y = 9$

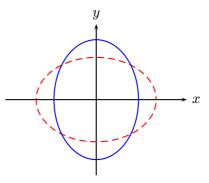
(f)
$$10x^2 = \sqrt{2 - 10y^2}$$

(g)
$$(x+12y)(x-12y) = -13$$

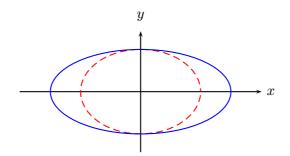
26.6 Find the foci and center of ellipse $\frac{(x-1)^2}{9} + \frac{(y-2)^2}{16} = 1$.

26.7 Find the x radius, y radius, foci, and center of $x^2 + 4y^2 - 6x + 8y = 3$.

26.8 On the next graph are two ellipses $\frac{x^2}{4} + \frac{y^2}{2} = 1$ and $\frac{x^2}{2} + \frac{y^2}{4} = 1$. Which curve represents $\frac{x^2}{4} + \frac{y^2}{2} = 1$?



26.9 On the next graph are two ellipses $\frac{x^2}{4} + \frac{y^2}{2} = 1$ and $\frac{x^2}{9} + \frac{y^2}{2} = 1$. Which ellipse represents $\frac{x^2}{4} + \frac{y^2}{2} = 1$?



26.10 Write an ellipse to satisfy each set of conditions.

- (a) The center is at the origin. Its two axes are 10 and 8 units long, and the major axis is on the y axis.
- (b) Shift $3x^2 + 4y^2 = 5$ horizontally by 3 units.
- **26.11** At most how many points can two different ellipses intersect at? How many points are necessary to determine an ellipse?

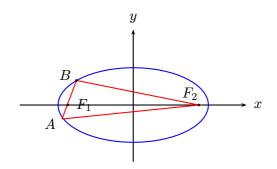
26.12 Find two points on $(x-1)^2 + \frac{(y-2)^2}{4} = 1$ that are farthest apart.

26.13 Write new equation after each operation on ellipse $\frac{(x-1)^2}{2^2} + \frac{(y-3)^2}{4^2} = 1.$

- (a) Rotate it clockwise around its center 90° .
- (b) Rotate it clockwise around the origin 90° .
- **26.14** Given two points A(1, 0) and B(-1, 0), find the path of a moving point P(x, y) such that |PA| + |PB| = 2.

[©] Qishen Huang

- **26.15** When a = b, ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a circle. As a special ellipse, a circle of radius r has area of πr^2 . Can you guess the formula for the area of a general ellipse?
- **26.16** The foci of ellipse $\frac{x^2}{4} + y^2 = 1$ are points F_1 and F_2 . Chord AB passes point F_1 . Find the circumference of $\triangle ABF_2$.



- **26.17** Point F_1 is the left focus of ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. *P* is a moving point on the ellipse. *Q* is the midpoint of *PF*₁. Find the trajectory of *Q*.
- **26.18** Line segment AB is a diameter of a circle. Point C is on the circle. Connect A with C, and B with C. We know from geometry that $|AB|^2 = |AC|^2 + |BC|^2$. Now we consider a genuine ellipse which is not a circle. If AB is the major axis of the ellipse, C is another point on the ellipse. Compare $|AB|^2$ with $|AC|^2 + |BC|^2$.
- **26.19** A small wheel $x^2 + (y+1)^2 = 1$ rolls alone the inside edge of a bigger circle $x^2 + y^2 = 4$. Find the path of point P(0, -1.5) on the small wheel.

Hyperbolas

- 27.1 How many axes of symmetry does a hyperbola have?
- 27.2 Write hyperbolas in standard form.
 - (a) $3x^2 4y^2 = 5$
 - (b) $3x^2 4y^2 = -5$
 - (c) $x^2 + 2x 4y^2 3y = 1$

27.3 Identify hyperbolas.

- (a) $x^2 2y^2 = 3$ (b) $x^3 - 2xy^2 = 3x$ (c) $x^2 - 2y^2 = 0$ (d) $-x^2 - 2y^2 + 3 = 0$ (e) $x^4 - y^4 = 1$ (f) $x^2 - 2x - 3y^2 = 5$
- **27.4** Find the necessary and sufficient condition for $ax^2 + by^2 = 1$ to be each of the following curves.
 - (A) An ellipse.
 - (B) A hyperbola.
- 27.5 Find the foci, vertices, axes of symmetry, and asymptotes of hyperbolas.

(a)
$$\frac{x^2}{2} - \frac{y^2}{3} = 1$$

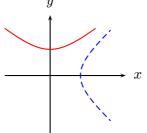
(b) $\frac{(x-2)^2}{2} - \frac{(y-3)^2}{3} = -1$

27.6 Write the equation of the left branch of hyperbola $\frac{(x-2)^2}{2} - \frac{(y-3)^2}{3} = 1.$

27.7 What is the graph of $y = 3 + \sqrt{x^2 - 4x}$?

- (a) A parabola.
- (b) A hyperbola.
- (c) An ellipse.
- (d) A branch of hyperbola.
- (e) None of above.

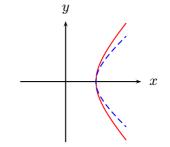
27.8 The next graph has two partial hyperbolas $\frac{x^2}{4} - \frac{y^2}{3} = 1$ and $-\frac{x^2}{4} + \frac{y^2}{3} = 1$. Identify the branch of $\frac{x^2}{4} - \frac{y^2}{3} = 1$.



27.9 How do we get graph of xy = -1 from that of xy = 1?

27.10 Express the set of points between the y axis and the right branch of $\frac{x^2}{4} - \frac{y^2}{9} = 1$ excluding the boundary.

27.11 The next graph has branches of two hyperbolas $\frac{x^2}{4} - \frac{y^2}{3} = 1$ and $\frac{x^2}{4} - \frac{y^2}{5} = 1$. Identify the branch of $\frac{x^2}{4} - \frac{y^2}{3} = 1$.



27.12 Determine the opening directions of hyperbolas.

(a) $3x^2 - 4y^2 = 5$ (b) $3x^2 - 4y^2 = -5$ (c) $\frac{(x-2)^2}{2} - \frac{(y-3)^2}{3} = 1$

27.13 Point (1, 6) is on hyperbola xy = k. Find the value of k.

- **27.14** If $a \neq b$, do the hyperbolas of xy = a and xy = b intersect?
- **27.15** Express the set of all points inside the left branch of hyperbola $\frac{x^2}{4} \frac{y^2}{9} = 1$.
- **27.16** Find all hyperbolas with asymptotes $y = \pm 2x$.
- **27.17** Among all hyperbolas with asymptotes $y = \pm 2x$, do any of them intersect?
- **27.18** Equation xy = 1 is a hyperbola. Give its center, vertices, asymptotes, and axes of symmetry.
- 27.19 Write the new equation after each rotation of hyperbola

$$\frac{(x-2)^2}{2} - \frac{(y-3)^2}{3} = 1.$$

- (a) Rotate it clockwise around its center by 90° .
- (b) Rotate it clockwise around the origin by 90° .
- **27.20** Rotate hyperbola xy = 1 clockwise around the origin by 45° . Write the equation of the new hyperbola.
- **27.21** How would you obtain graph of xy = 1 from that of $y = \frac{x-2}{2x-3}$?
- **27.22** Define a tangent line of a hyperbola as follows: It intersects only once with a branch of the hyperbola and does not cross the branch.

For hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$, any tangent line passes point (0, 0)?

- **27.23** Hyperbola $x^2 y^2 = 1$ has a left and a right branch. If its left branch is shifted horizontally by 0.5, do the two branches still form a hyperbola?
- **27.24** The distance between a moving point P(x, y) and (-2, 0) is twice that between P and line x = -1. Find the trajectory of P.
- **27.25** Given two points A(-1, 0) and B(1, 0) on the x axis, a moving point P(x, y) satisfies |PA| |PB| = 1. Write an equation for P.

[©] Qishen Huang

27.26 How do we intersect a cone with a plane to get the following figures?

- (a) A branch of a hyperbola.
- (b) Two lines.
- 27.27 Identify shapes (point, line, line segment, conic section) represented by the equations.
 - (a) $x^2 + 3y^2 = 0$
 - (b) (x-3y)(3x-y) = 0
 - (c) $x^3 xy = 0$
 - (d) $x^3 xy^2 = x$

Sequences: Basic

- **28.1** Review formulas of arithmetic sequence $\{a_1, a_2, a_3, ...\}$ with common difference of d. Assume m and n are positive integers.
 - $a_n = a_1 + (n-1)d$ $a_n = a_m + (n-m)d$ If m + n = p + q, then $a_m + a_n = a_p + a_q$. $S_n = \frac{n(a_1 + a_n)}{2}$
- **28.2** Review formulas of geometric sequence $\{a_1, a_2, a_3, ...\}$ with common ratio r. Assume m and n are positive integers.

$$a_n = a_m r^{n-m}$$

If $m + n = p + q$, then $a_m a_n = a_p a_q$
 $(\prod_{1}^n a_i)^2 = (a_1 a_n)^n$
 $S_n = \frac{a_1 - ra_n}{1 - r}$, where $r \neq 1$.

28.3 Express the general term in each of the following sequences.

(a) 1, 2, 3, 4, ... (b) 1, 2, 4, 8, 16, 32, ... (c) $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ (d) $-\frac{1}{1}, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots$ (e) $-\frac{1}{1}, \frac{2}{2}, -\frac{4}{3}, \frac{8}{4}, \dots$

© Qishen Huang

86

- (f) 1, 1, 2, 2, 3, 3, 4, 4, \dots
- (g) $0, 1, 0, 1, 0, 1, \ldots$
- **28.4** Is the statement correct? Sequence $\{a_n, n = 1, 2, 3, ...\}$ is equivalent to a function f with domain of positive integers and $f(n) = a_n$.
- **28.5** Find the 168th term of arithmetic sequence $\{a_n, n = 1, 2, 3, ...\}$ with common difference d = 1.5 and first term $a_1 = 2$.
- **28.6** Find the 8th term of geometric sequence $\{a_n, n = 1, 2, 3, ...\}$ with common ratio r = 0.5 and first term $a_1 = 2$.
- **28.7** The arithmetic mean of a and b is defined as $\frac{a+b}{2}$. Prove that in an arithmetic sequence, a term is the arithmetic mean of its two adjacent terms.
- **28.8** Given a > 0 and b > 0, their geometric mean is defined as \sqrt{ab} . Prove that in a geometric sequence $\{a_n\}$ of only positive terms, a term is the geometric mean of its two adjacent terms.
- **28.9** Arithmetic sequence $\{a_n, n = 1, 2, 3, ...\}$ has common difference of 2. Compute the values of $a_3 a_1$ and $a_{100} a_{50}$.
- **28.10** In arithmetic sequence $\{a_1, a_2, a_3, \dots\}, a_1 + a_4 = 10$. Find the value of $a_2 + a_3$.
- **28.11** In arithmetic sequence $\{a_1, a_2, a_3, \dots\}, a_9 + a_{11} = 10$. Find the value of a_{10} .
- **28.12** In arithmetic sequence $\{a_1, a_2, a_3, \dots\}, a_4 a_1 = 12$. Find its common difference.
- **28.13** In geometric sequence $\{a_1, a_2, a_3, \dots\}, \frac{a_4}{a_1} = 10$. Find its common ratio.
- **28.14** In geometric sequence $\{a_1, a_2, a_3, ...\}, a_1a_4 = 10$. Find the value of a_2a_3 .
- **28.15** Two arithmetic sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_9 = b_9$ and $a_{19} = b_{19}$. Prove $a_n = b_n$ for all integer $n \ge 1$.
- **28.16** Two geometric sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_9 = b_9$ and $a_{20} = b_{20}$. Prove $a_n = b_n$ for any positive integer n.
- **28.17** Sequence $\{a_i\}$ is an arithmetic sequence. Suppose $a_m > a_n$ for some integers m and n satisfying m > n. Prove that the sequence is increasing.
- **28.18** If values of two terms of a geometric sequence are in interval [a, b], can any term located between them in the sequence have a value outside [a, b]?
- **28.19** If terms a_3 and a_{10} of a geometric sequence $\{a_n, n = 1, 2, 3, ...\}$ are positive, can any other term be negative?

- **28.20** Given sequence $\{a_i = i\}$, list first five terms of the following derived sequences. Also express b_{n-1} and b_{n+1} in terms of a_i .
 - (a) $\{b_n = a_{n+2}\}$
 - (b) $\{b_n = a_{2n}\}$
 - (c) $\{b_n = 4 + a_n\}$
 - (d) $\{b_n = a_{2n+2} a_{2n}\}$
 - (e) $\{b_n = \frac{1}{a_n}\}$
- **28.21** Reverse a finite arithmetic sequence $\{a_1, a_2, \ldots, a_9\}$ with common difference of 1 to get a new sequence $\{b_i = a_{9-i}, i = 0, 1, 2, \ldots, 8\}$. Is the new sequence arithmetic? If yes, find its common difference.
- **28.22** Reverse a finite geometric sequence $\{a_1, a_2, \ldots, a_9\}$ with common ratio of 2 to get a new sequence $\{b_i = a_{9-i}, i = 0, 1, 2, \ldots, 8\}$. Is the new sequence geometric? If yes, find its common ratio.
- **28.23** If a term a_m , where m > 1, is removed from infinite arithmetic sequence $\{a_n\}$ with common difference of 2, is the new sequence still an arithmetic sequence?
- **28.24** If value 1 is added to a term a_m of geometric sequence $\{a_n\}$ with common ratio of 2, is the new sequence still a geometric sequence?
- **28.25** In geometric sequence $\{a_1, a_2, a_3, \dots\}, a_9a_{11} = 10$. Find the value of $|a_{10}|$.
- **28.26** Give an example of a geometric sequence $\{a_1, a_2, a_3, ...\}$ that is also an arithmetic sequence.
- **28.27** If sequence $\{a_n\}$ is arithmetic with non-zero common difference of d, which of the following derived sequences are arithmetic? Find the common differences if applicable.
 - (a) $\{b_n = ka_n + h\}$, where k and h are some constant.
 - (b) $\{b_n = a_{7+n}\}$
 - (c) $\{b_n = a_n^2\}$
 - (d) $\{b_n = |a_n|\}$
 - (e) $\{b_n = a_{2n+1}\}$
 - (f) $\{b_n = a_{n+k} a_n\}$, where k is some constant.
- **28.28** Sequence $\{a_n\}$ is a geometric sequence with non-zero common ratio r. Which of following derived sequences are geometric? Find the common ratios if applicable.
 - (a) $\{b_n = ka_n\}$, where k is some constant.

[©] Qishen Huang

- (b) $\{b_n = a_{2n}\}$
- (c) $\{b_n = a_n^2\}$
- (d) $\{b_n = |a_n|\}$
- (e) $\{b_n = a_{9+n}\}$

(f)
$$\{b_n = \frac{1}{a_n}\}$$

- (g) $\{b_n = \frac{a_{n+k}}{a_n}\}$, where k is some constant.
- **28.29** If $\{a_n\}$ and $\{b_n\}$ are arithmetic sequences with common differences of d_1 and d_2 respectively, is $\{a_n + b_n\}$ an arithmetic sequence too? If yes, find its common difference.
- **28.30** If $\{a_n\}$ and $\{b_n\}$ are geometric sequences with common ratios of r_1 and r_2 , is $\{a_nb_n\}$ a geometric sequence too? If yes, what is the common ratio?
- **28.31** Given a geometric sequence $\{a_i = ar^{i-1}, a > 0, r > 0, i = 1, 2, 3, ...\}$, prove that $\{b_i = \log a_i, i = 1, 2, 3, ...\}$ is an arithmetic sequence.
- **28.32** Arithmetic sequence $\{a_i, i = 1, 2, 3, ...\}$ has common difference of d. Prove that $\{b_i = e^{a_i}, i = 1, 2, 3, ...\}$ is a geometric sequence.
- 28.33 Identify true statements.
 - (a) A sequence is either arithmetic or geometric.
 - (b) An arithmetic sequence is either increasing or decreasing if common difference is not zero.
 - (c) A geometric sequence is either increasing or decreasing if common ratio is not 1.
 - (d) An infinite arithmetic sequence is not bounded if its common difference is not zero.
 - (e) A geometric sequence can be bounded or unbounded.
 - (f) If first few terms are dropped from an arithmetic (geometric) sequence, the new sequence is still arithmetic (geometric) with common difference (ratio) unchanged.
 - (g) If the last few terms are dropped from a finite arithmetic (geometric) sequence, the new sequence is still arithmetic (geometric) with common difference (ratio) unchanged.
 - (h) Given any geometric sequence with non-zero common ratio and non-zero first term, the two sub-sequences $\{a_{2n-1}\}$ and $\{a_{2n}\}$ are monotonic (increasing or decreasing).

- **28.34** In sequence $\{a_n\}$, $a_1 = 10$ and $a_n = a_{n-1} + 2$, where n > 1. Express a_n in terms of n.
- 28.35 The sum of the first five terms of an arithmetic sequence is 20. Find the third term.
- **28.36** The product of the 7th, 8th, and 9th term of a geometric sequence is 27. Find the 8th term.
- **28.37** Given geometric sequence $\{a_n = q^n\}$ where $q \neq 1$, derive a formula for

 $a_2 + a_4 + a_6 + \dots + a_{2n}$.

- **28.38** In sequence $\{a_i\}$, $a_1 = 0$ and $a_n = a_{n-1} + 2n$, where $n \ge 2$. Find the value of term a_{100} .
- **28.39** Compute sum of $2 + 2^4 + 2^7 + \dots + 2^{3n+1}$.

Sequences: Intermediate

29.1 Prove the property of arithmetic sequence $\{a_i\}$:

If
$$m + n = p + q$$
, then $a_m + a_n = a_p + a_q$

29.2 Prove the property of arithmetic sequence $\{a_i\}$:

$$S_n = \frac{n(a_1 + a_n)}{2}.$$

29.3 Prove the property of geometric sequence $\{a_i\}$:

If m + n = p + q, then $a_m a_n = a_p a_q$.

.

29.4 Prove the property of geometric sequence $\{a_i\}$:

$$\left(\prod_{1}^{n} a_i\right)^2 = (a_1 a_n)^n$$

29.5 Prove the property of geometric sequence $\{a_i = ar^{i-1}, i = 1, 2, 3, ...\}$:

$$S_n = \frac{a_1(1-r^n)}{1-r}$$
, where $r \neq 1$.

29.6 We know the property of geometric sequence $\{a_i = ar^{i-1}\}$:

$$S_n = \frac{a(1-r^n)}{1-r}$$
, where $r \neq 1$.

What is S_n if r = 1?

29.7 The 10th and the 20th term of an arithmetic sequence are 10 and 40, respectively. Find its 100th term.

- **29.8** The 10th and the 12th term of a geometric sequence are 10 and 12.1, respectively. Find its 14th term.
- **29.9** Solve for integer *n* in inequality $1 + 3^1 + 3^2 + \dots + 3^n > 100$.
- **29.10** For sequence $\{a_n\}$, $S_n = na + n(n-1)b$, where a and b are some constants. Find algebraic form of a_n . Is the sequence arithmetic, geometric, or neither?
- **29.11** In arithmetic sequence $\{a_n, n = 1, 2, 3, ...\}, a_1 = 2$ and $a_2 + a_3 = 13$. Compute the sum of $a_4 + a_5 + a_6$.
- **29.12** In arithmetic sequence $\{a_n, n = 1, 2, 3, ...\},\$

 $a_1 + a_3 + a_5 + a_7 + a_9 = 15$

and

 $a_2 + a_4 + a_6 + a_8 + a_{10} = 30.$

Find its common difference.

- **29.13** In sequence $\{a_n, n = 1, 2, 3, ...\}$, $a_1 = 1$ and $a_{n+1} = 2a_n + 3$, where $n \ge 1$. Express a_n in terms of n.
- **29.14** A geometric sequence has common ratio $r \neq 0$ and first term $a_1 \neq 0$. Discuss $\lim_{n \to \infty} a_n$ in each case.
 - (a) r > 1
 - (b) r = -1
 - (c) |r| < 1

Sequences: Advanced

- **30.1** Express rational value $0.\overline{19}$ as a fractional number in the simplest form.
- **30.2** Derive a formula for sum of $\sum_{i=1}^{n} i^2$.
- **30.3** Compute the value of $\sum_{n=1}^{80} (-1)^n n^2$.
- **30.4** Solve inequality $\sum_{i=1}^{n} \left(\frac{1}{2}\right)^{i} > 0.99.$
- **30.5** Derive a formula for $S_n = 1 + 101 + 10101 + \dots + \underbrace{10101\dots01}^{2n-1}$, where $n \ge 1$.
- **30.6** Given arithmetic sequence $\{a_n, n = 1, 2, 3, ...\}$, define another sequence $\{b_n = |a_n|\}$. In general, $\{b_n\}$ is not an arithmetic sequence. Prove that if the terms are all non-negative or all non-positive, it an arithmetic sequence.
- **30.7** In an increasing arithmetic sequence $\{a_n, n = 1, 2, 3, ...\}$,

 $a_1a_2a_3 = 80$ and $a_1 + a_2 + a_3 = 15$.

Find the common difference.

30.8 Given function $f(x) = x^2 + 2x$, define sequence $\{a_n, n = 1, 2, 3, ...\}$ as follows:

 $a_1 = 2, a_2 = f(a_1), a_3 = f(a_2), and so forth.$

Prove that sequence $\{\log(1 + a_n)\}$ is a geometric sequence.

- **30.9** A finite arithmetic sequence of positive integers starts at 3 and ends at 49. Another term between them is 25. Find the index of the term 25.
- **30.10** Find the real roots of equation $1 + x + x^2 + \cdots + x^5 = 0$.
- **30.11** Compute the minimum value of function $f(x) = \sum_{i=1}^{19} |x-i|$.

Trigonometry: Basic

31.1 Review important identities.

- (a) $\sin^2 x + \cos^2 x = 1$
- (b) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
- (c) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

(d)
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

(e)
$$\sin(-x) = -\sin x$$

(e)
$$\sin(-x) = -\sin x$$

(f) $\cos(-x) = \cos x$

(f)
$$\cos(-x) = \cos x$$

(g)
$$\tan(-x) = -\tan x$$

(h)
$$\sin(2x) = 2\sin x \cos x$$

(i)
$$\cos(2x) = \cos^2 x - \sin^2 x$$

(j)
$$\tan(2x) = \frac{2\tan x}{1-\tan^2 x}$$

(k) $\sin x \pm \sin y = 2\sin \frac{x \pm y}{2} \cos \frac{x \mp y}{2}$

(1)
$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

(m)
$$\cos x - \cos y = 2\sin \frac{x+y}{2}\sin \frac{x-y}{2}$$

(n)
$$\sin x \sin y = -\frac{1}{2} \left[\cos(x+y) - \cos(x-y) \right]$$

(o)
$$\cos x \cos y = \frac{1}{2} [\cos(x+y) + \cos(x-y)]$$

(p)
$$\sin x \cos y = \frac{1}{2} [\sin(x+y) + \sin(x-y)]$$

31.2 Recognize common mistakes. These are generally false.

- (a) $\sin(x \pm y) = \sin x \pm \sin y$
- (b) $\cos(x \pm y) = \cos x \pm \cos y$
- (c) $\tan(x \pm y) = \tan x \pm \tan y$
- (d) $\sin(k\alpha) = k\sin\alpha$
- (e) $\sin(xy) = (\sin x)(\sin y)$
- **31.3** Two triangles, $\triangle ABC$ and $\triangle A_1B_1C_1$, are different except $A = A_1$. Is $\sin A = \sin A_1$?

31.4 Give trigonometric values without a calculator.

- (a) $\sin 45^{\circ}$
- (b) $\cos 60^{\circ}$
- (c) $\tan 30^{\circ}$
- (d) $\cot 0^{\circ}$
- (e) $\sin 90^{\circ}$
- (f) $\cos 180^{\circ}$
- (g) $\cos 210^{\circ}$
- (h) $\tan 300^{\circ}$
- (i) $\tan 660^{\circ}$

31.5 Simplify expressions.

- (a) $\sin(x + \frac{\pi}{2})$ (b) $\sin(\frac{\pi}{2} - x)$ (c) $\sin(x + \pi)$ (d) $\sin(\pi - x)$ (e) $\cos(x + \frac{\pi}{2})$ (f) $\cos(\frac{\pi}{2} - x)$ (g) $\cos(x + \pi)$ (h) $\cos(\pi - x)$
- (II) $\cos(\pi x)$
- (i) $\sin(2\pi x)$
- (j) $\cos(2\pi x)$
- (k) $\tan(2\pi x)$

(1)
$$\sin(x + \frac{2}{3}\pi)$$

(m) $\sin(\frac{2}{3}\pi - x)$
(n) $\cos(x + \frac{2}{3}\pi)$
(o) $\cos(\frac{2}{3}\pi - x)$
(p) $\tan(x + \frac{2}{3}\pi)$
(q) $\tan(\frac{2}{3}\pi - x)$

31.6 Define set A as

$$A = \{ n\pi + (-1)^n \frac{\pi}{2}, \ n \in Z \}.$$

List the terms for n = 0, 1, 2, and 3.

31.7 Define five sets:

$$A = \{x \mid x = 2n\pi + \frac{\pi}{2}, n \in Z\},\$$
$$B = \{x \mid x = 2n\pi - \frac{\pi}{2}, n \in Z,\$$
$$C = \{x \mid x = k\pi \pm \frac{\pi}{2}, k \in Z\},\$$
$$D = \{x \mid x = 2n\pi \pm \frac{\pi}{2}, n \in Z\},\$$
$$E = \{x \mid x = k\pi + \frac{\pi}{2}, k \in Z.\$$

Which statements are true?

- (a) C = D(b) $A \cup B = C$ (c) $A \subset E$
- **31.8** Angles α and β are coterminal. Is $\alpha \beta$ a multiple of 2π ?

31.9 Express the set of all angles with terminal side on any axis.

31.10 Find all angles that are coterminal with -263° .

31.11 Find the angles between 2π and 3π that are coterminal with $\frac{7}{8}\pi$.

- **31.12** The terminal side of θ is in the 1st quadrant. Additionally, $\cos \theta = \frac{4}{5}$. Find the value of $\tan \theta$.
- **31.13** The terminal side of angle θ is in the 4th quadrant. In addition, $\cos \theta = \frac{4}{5}$. Find the value of $\tan \theta$.
- **31.14** If $x \in [a, b]$, which expression is true?

(A)
$$\frac{x}{2} \in [\frac{a}{2}, \frac{b}{2}]$$

(B) $\frac{x}{2} \in [2a, 2b]$

31.15 If terminal side of angle α is in the first quadrant, where are those of 2α and $\frac{\alpha}{2}$? **31.16** Find the quadrant the terminal side of angle θ is in.

(a)
$$\theta = -3$$
.
(b) $\theta = \frac{128\pi}{17}$.
(c) $\sin \theta < 0$ and $\cos \theta < 0$.

31.17 If two angles x and y of a triangle satisfy $\sin x = \sin y$, are they equal?

31.18 If two angles x and y satisfy $\sin x = \sin y$ and $\cos x = \cos y$, are they coterminal?

31.19 Compute without using a calculator.

(a)
$$\sin \frac{9\pi}{4} \tan \frac{7\pi}{3}$$

(b) $\cos^4 \frac{\pi}{6} - \sin^4 \frac{\pi}{6}$
(c) $\tan 10^\circ \cdot \tan 80^\circ$
(d) $2\sin 765^\circ + \tan 1485^\circ \cot 135^\circ$

31.20 Give the domains and ranges of the functions.

(a)
$$y = \sin x$$

(b) $y = \cos x$
(c) $y = \tan x$
(d) $y = \cot x$

31.21 Solve three equations and an inequality.

(a) $\cos x = 0$

[©] Qishen Huang

- (b) $\tan x = 1$
- (c) $\sin x \cdot (1 \sin x) = 0$
- (d) $0 < \tan x < 1$
- **31.22** We know that π is an irrational number. How many angles of a triangle can be rational in radians?
- **31.23** The minute hand of an accurate clock is twice as long as the hour hand. Their speeds are measured in degrees. Find the ratio of the two speeds. If measured in radians, would the ratio be different?
- **31.24** Identity $\sin(90^\circ x) = \cos x$ is true for any acute angle x. Is it true for any angle x?
- **31.25** John remembers that

 $\cos(x+y) = \cos x \cos y + \sin x \sin y \quad \text{or} \quad \cos x \cos y - \sin x \sin y.$

He is not sure about the operator, + or -, on the right hand sides. How would you find out using some special values of x and y?

- **31.26** Use the values of sine and cosine of 30° and 45° to derive the value of $\sin 15^{\circ}$.
- **31.27** Find the value of $\tan \frac{\pi}{12}$.
- **31.28** Prove that $\cos 2x \le \cos^2 x$.
- **31.29** By the Law of Sines, in $\triangle ABC$,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

What is the geometric interpretation of $\frac{a}{\sin A}$?

- **31.30** In $\triangle ABC$, a = 2, b = 3, and c = 4. Find the value of $\cos C$. Is the triangle acute, right, or obtuse?
- **31.31** In a general $\triangle ABC$, which identities are true?
 - (a) $\sin A = \sin(B+C)$
 - (b) $\cos A = \cos(B+C)$
 - (c) $\tan A = \tan(B+C)$

31.32 Determine the value of side a of $\triangle ABC$ with $B = 30^{\circ}$, b = 12, and c = 24.

31.33 In a right $\triangle ABC$ with $C = 90^{\circ}$,

$$a^2 + b^2 = c^2.$$

In a general triangle,

 $a^2 + b^2 - 2ab\cos C = c^2.$

Are these two formulas consistent?

- **31.34** Given three sides a, b, and c of $\triangle ABC$, how do you find the measure of angle A?
- **31.35** Given A, B, and c of $\triangle ABC$, how do you find the two other sides?
- **31.36** Given a, b, and C of $\triangle ABC$, how do you find the other two angles and the third side?
- **31.37** Given two sides a and b of $\triangle ABC$, what angle C maximizes the area of the triangle?
- **31.38** In $\triangle ABC$, c = 1 and $C = 15^{\circ}$. Determine the diameter of its circumscribed circle.
- **31.39** Is this function a sinusoid: $y = A\cos(b(x-h)) + k$?

31.40 Find the amplitude, period, and phase of function $y = 2\sin(\frac{x}{3} - \frac{\pi}{4})$.

- **31.41** A sinusoid with amplitude of 3 has maximum of 4. Find its minimum value.
- **31.42** A periodical function f(x) with period of 30 passes point (1, 2). Find the value of f(61).
- **31.43** Function $f(x) = \tan(ax + b)$, where a and b are some constants. Line x = 5 is an asymptote of f. Find the value of function $g(x) = \cot(ax + b)$ at 5.
- **31.44** Function $f(x) = -x \cos x$. Determine whether it has the properties below.
 - (a) It is an odd function.
 - (b) It is bounded.
 - (c) It is a one-to-one mapping.
 - (d) It is a periodical function.
 - (e) It is continuous.
- **31.45** How do you get the graph of $\cos x$ from that of $\sin x$?
- **31.46** How would you obtain the graph of $\cot x$ from that of $\tan x$?

31.47 How does one obtain the graph of $y = \sin(3x + \frac{\pi}{6})$ from that of $y = \sin(3x)$?

[©] Qishen Huang

- **31.48** We shrink the graph of function $y = 2\sin(2x + \frac{\pi}{3})$ vertically by a half. Write the equation of the new graph.
- **31.49** We stretch the graph of $f(x) = 2\sin(2x + \frac{\pi}{3})$ horizontally in both directions by a factor of 100%. Write the new function.
- **31.50** Identify even functions.

(a)
$$y = \sin |x|$$

(b) $y = \sin(x - \frac{17\pi}{2})$
(c) $y = \sin^2 x + x \sin x + 5$

(d)
$$y = \tan(x - \frac{\pi}{2})$$

31.51 In $\triangle ABC$, find A in radians under each condition. Use a calculator if necessary.

- (a) $\sin A = 0.25$
- (b) $\cos A = 0.25$
- (c) $\tan A = 0.25$
- (d) $\cos B = 0.25$, $\cos C = 0.52$

31.52 Given $x \in (\frac{\pi}{2}, \pi)$ and $\sin x = \frac{\sqrt{3}}{3}$, find the value of x.

31.53 State the domain and range of the following inverse functions.

(a)
$$y = \sin^{-1}(x)$$

(b) $y = \cos^{-1}(x)$
(c) $y = \tan^{-1}(x)$

31.54 Remove negative sign before x from the inverse functions without changing value.

(a) $\sin^{-1}(-x)$ (b) $\cos^{-1}(-x)$ (c) $\tan^{-1}(-x)$

31.55 Which statements are always true?

(a)
$$\sin^{-1}(2x) = 2\sin^{-1}x$$

(b) $\sin^{-1}(2x) = (\sin^{-1}x)/2$
(c) $\sin^{-1}(2x) = 2 + \sin^{-1}x$

- (d) $\sin^{-1}(2x) = \sin^{-1}2 + \sin^{-1}x$
- (e) $\sin^{-1}(2x) = \sin^{-1} 2 \cdot \sin^{-1} x$

31.56 Define three functions as follows:

$$f(x) = \sin x, \ x \in R,$$

$$g(x) = \sin x, \ x \in [-\frac{\pi}{2}, \frac{\pi}{2}],$$

$$h(x) = \sin^{-1} x, \ x \in [-1, 1]$$

What is the difference between f and g? Is h the inverse function of f or g?

31.57 Compute values without a calculator.

(a)
$$\sin^{-1}(\cos\frac{4}{5}\pi)$$

(b) $\sin^{-1}(\sin 2)$
(c) $\sin^{-1}(\sin 4\pi)$

31.58 Which statements are always true?

(a) sin (sin⁻¹ x) = x, where x ∈ [-1, 1].
(b) cos⁻¹(cos x) = x, where x ∈ R.

31.59 Find the domain and inverse function of $f(x) = \cos^{-1}(x-3)$.

- **31.60** Find the inverse function of $y = \sin x$, where $x \in (\frac{3\pi}{2}, 2\pi)$.
- **31.61** Find the domain, minimum, and maximum of $y = \sin^{-1}(2x 1)$.

31.62 Without a calculator, determine whether the following expressions are positive.

(a)	$\cos^{-1}(-\frac{3}{5}) - \cos^{-1}(-\frac{1}{5})$
(b)	$\tan^{-1}(\frac{3}{4}) - \tan^{-1}(\frac{3}{8})$

31.63 Solve equation $\cos x = -\frac{2}{3}$, where $x \in [0, \pi]$. **31.64** Solve equation $\cos x = \frac{1}{5}$, where $x \in [3\pi, 4\pi]$.

31.65 Solve $3\tan(\frac{x}{2} + \frac{\pi}{12}) = -\sqrt{3}$.

31.66 Can a non-constant periodical function on R or its subset have these properties?

[©] Qishen Huang

- (a) increasing or decreasing
- (b) invertible
- (c) unbounded

31.67 State the periods of the following functions.

- (a) $f(x) = \sin(x)$
- (b) $g(x) = \cos(x)$
- (c) $h(x) = \tan(x)$

Is the period of h equal to those of f and g?

31.68 Find the periods of the following functions.

(a)
$$\cos^2 x$$

(b) $\frac{\sin 2x}{\cos x}$
(c) $4\sin(\frac{\pi}{4} - x)$
(d) $f(x) = \sin x - |\sin x|$

31.69 Identify odd and even functions.

(a)
$$f(x) = \sin(x + \frac{3}{5}\pi)$$

(b) $f(x) = |\sin x| + \cos x$
(c) $f(x) = \sin(\pi + x)\cos(\frac{\pi}{2} + x)$

31.70 Find the maximum and minimum of $y = \sin x \cos x$.

Trigonometry: Intermediate

32.1 If $\sin 2\alpha > 0 > \cos \alpha$, which quadrants can the terminal side of angle $\frac{\alpha}{2}$ be in?

32.2 Are the inequalities true? Don't use a calculator.

- (a) $\sin 3 > \sin 4$
- (b) $\cos 3 > \cos 4$

32.3 Given $\sin(\frac{3\pi}{2} - x) = -\frac{4}{5}$, compute the value of $\cos x$.

- **32.4** Prove that $\sin x < x$ if $\frac{\pi}{2} > x > 0$.
- **32.5** Suppose $\theta \in (-\frac{\pi}{2}, 0)$ and $\cos \theta = \frac{3}{5}$. Find the value of $\sin(\theta \frac{\pi}{6})$.
- **32.6** Derive identity $\sin(3x) = 3\sin x 4\sin^3 x$.
- **32.7** Derive identity $\cos 3x = -3\cos x + 4\cos^3 x$.
- **32.8** Simplify expressions so the arguments in trigonometric functions are as simple as possible.

(a)
$$\sqrt{\frac{1+\cos 260^{\circ}}{2}}$$

(b) $2\cos(\frac{\pi}{4}+\alpha)\cos(\frac{\pi}{4}-\alpha)$

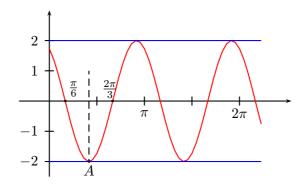
32.9 Given $180^{\circ} > \alpha > 90^{\circ}$ and $\sin \alpha = \frac{4}{5}$, compute the value of $\tan \frac{\alpha}{2}$.

32.10 Find the range of $f(x) = e^{\tan^{-1} x}$, where $x \in R$.

32.11 Prove identity $\tan 3x - \tan 2x - \tan x = \tan 3x \tan 2x \tan x$.

[©] Qishen Huang

- **32.12** Acute angle x satisfies $\tan^2 x \tan x = 6$. Find the value of x.
- **32.13** Without a calculator, compute $\sin^{-1}(-\frac{3}{5}) + \cos^{-1}(-\frac{3}{5})$.
- **32.14** Without a calculator, compute $\tan^{-1} 1.2 + \cot^{-1} 1.2$.
- **32.15** Compute the value of $\sin \frac{5\pi}{12}$ without a calculator.
- **32.16** Given $\sin x + \cos x = \sqrt{2}$, compute the value of $\sin x \cos x$.
- **32.17** Prove that the area of $\triangle ABC$ is $\frac{a^2 \sin B \sin C}{2 \sin A}$.
- **32.18** Simplify $f(x) = \tan(|\tan^{-1} x|)$.
- **32.19** Find a line that does not intersect with $\sin x$.
- **32.20** Where does function $y = \sin x$ intersect function $y = \tan x$?
- **32.21** Assume $\theta \in (0, \frac{\pi}{4})$. Compare the values of $\sin \theta$, $\cos \theta$, and $\cot \theta$.
- **32.22** Find the number of solutions of sin(10x) = 0.5, where $x \in [0, \pi]$.
- **32.23** The curve is the graph of $f(x) = a \sin k(x \beta)$ where a, k, and β are some constants. Find the smallest positive values of the three constants.



32.24 Compute values without a calculator.

(a)
$$\cos(\sin^{-1}x), x \in [-1, 1]$$

- (b) $\sin(\cos^{-1}x), x \in [-1, 1]$
- (c) $\sin^{-1} 0.5 + \cos^{-1} 0.5$
- (d) $\sin^{-1}(\frac{\pi}{5}) + \cos^{-1}(\frac{\pi}{5})$

32.25 Identify periodical functions and state their periods.

- (a) $y = \sin |x|$
- (b) $y = |\sin 2x|$
- (c) $y = \sin^2 x \cos^2 x$
- (d) $y = \sin^4 x + \cos^4 x$

(e)
$$f(x) = 2\sin(x + \frac{\pi}{3})\sin(x + \frac{\pi}{2})$$

32.26 Find the minimum of $f(x) = \sin x - \sqrt{3} \cos x$, where $x \in R$.

32.27 Is function
$$y = \cos(\frac{13\pi}{2} - 2x)$$
 increasing on each $[k\pi - \frac{\pi}{4}, k\pi + \frac{\pi}{4}]$, where $k \in \mathbb{Z}$?

Trigonometry: Advanced

33.1 Find the maximum value of $\sin x \cdot (\sin x + \cos x)$.

- **33.2** Simplify $\sin 8^{\circ} \cos 52^{\circ} \sin 37^{\circ} \cos 7^{\circ}$.
- **33.3** Solve $\tan^{-1} x + \tan^{-1}(x-1) = \tan^{-1} 3$.
- **33.4** Solve equation $\sqrt{1 \cos^2 x} = \cos x$.
- **33.5** Find the smallest positive solution of $5\sin^2 x 2\cos x = 1$.

33.6 Compute values without a calculator.

(a) $\sin 110^{\circ} \sin 140^{\circ} + \sin 40^{\circ} \sin 70^{\circ}$

(b)
$$\cos 60^{\circ} - \cos 40^{\circ} + 2\cos 80^{\circ} \cos 160^{\circ}$$

(c)
$$\frac{\cot 70^{\circ} \cot 140^{\circ} - 1}{\cot 70^{\circ} - \tan 50^{\circ}}$$

33.7 Simplify the expression to eliminate the radical signs:

$$\sqrt{1 + \sqrt{\frac{1}{2} - \frac{1}{2}\cos 2x}}, \quad x \in (\frac{3\pi}{2}, 2\pi).$$

33.8 Compute $\sum_{n=0}^{9} \sin \frac{2n\pi}{9}$.

- **33.9** If $\sin x + \cos x = 0.8$, compute the value of $\sin^4 x + \cos^4 x$.
- **33.10** Two lines y = x + 1 and y = 2x + 1 cross each other and are symmetric with respect to two other lines. Find the slopes of the two axes of symmetry.

- **33.11** Determine the type of $\triangle ABC$ under each of following conditions. Be as specific as possible. For instance, isosceles triangle is a wrong answer for an isosceles right triangle.
 - (1) $\sin A = \sin B$
 - (2) $\sin A : \sin B : \sin C = 3 : 4 : 5$
 - (3) $\sin A \sin B = \cos A \cos B$
 - $(4) \quad \sin(2A) = \sin(2B)$
 - (5) $\cos A \cos B \cos C > 0$
- **33.12** $\triangle ABC$ is an acute triangle. Prove that $a \cos B + b \cos A = c$.

33.13 In $\triangle ABC$, $C = 90^{\circ}$. Find the maximum of $\sin A + \sin B$.

33.14 Use the Law of Sines to prove the theorem:

In
$$\triangle ABC$$
, if $A > B$, then $a > b$.

- **33.15** Solve inequality $\sin x \cos x < 0$, where $x \in [0, 2\pi)$.
- **33.16** Function $y = A\sin(ax + \omega)$, where A > 0 and $\omega > 0$. It has minimum of $-\frac{1}{2}$ at $x = \frac{4\pi}{9}$ and maximum at $x = \frac{\pi}{9}$. Find its algebraic form.
- **33.17** The graph of sinusoid function $y = \sin x$ cuts a straight line into many equal segments. Find the general form of the line.
- **33.18** Can a straight line cross the graph of $y = \tan x$ where $x \in R$ exactly once, twice, infinite times, or none?
- **33.19** Compute the values without a calculator.
 - (a) $\tan(\tan^{-1} 1.8 + \pi)$ (b) $\tan(\sin^{-1} 0.6)$ (c) $\cos(\frac{1}{2}\sin^{-1}(-\frac{4}{5}))$ (d) $\tan^{-1}(-2) + \tan^{-1}(-3)$

33.20 Find the smallest positive solution of equation $\sin(2x + \pi/4) = \cos(\pi/6 - x)$.

33.21 Solve inequality $\sin^{-1} x \ge \cos^{-1} x$.

33.22 Prove identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, where $x \in [-1, 1]$.

33.23 Solve equation $\sin 3x = \sin 2x + \sin x$, where $x \in [0, 2\pi)$.

[©] Qishen Huang

- **33.24** The period of non-constant periodical function f(x) is a. Which of the following functions are periodical? If applicable, find the periods.
 - (a) g(x) = kf(x) + b, where $k \neq 0$.
 - (b) g(x) = f(kx + b), where $k \neq 0$.
 - (c) $g(x) = (f(x))^3$.
 - (d) g(x) = |f(x)|.
 - (e) g(x) = f(|x|).

33.25 Find periods of the following functions.

(a)
$$f(x) = \cos 2x + \sin(\frac{\pi}{2} + x).$$

(b) $f(x) = \sqrt{1 - \cos 2x} + \sqrt{1 + \cos 2x}.$
(c) $f(x) = \frac{\cos 2x + \sin 2x}{\cos 2x + \sin 2x}$ where $x \neq \frac{k}{2}\pi + \frac{\pi}{2}$ and

(c)
$$f(x) = \frac{\cos 2x + \sin 2x}{\cos 2x - \sin 2x}$$
, where $x \neq \frac{\pi}{2} + \frac{\pi}{8}$ and $k \in \mathbb{Z}$.

33.26 Find the maximum of function $y = (1 + \sin x)(1 + \cos x)$.

Complex Numbers

- **34.1** Assume $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$. Review formulas of complex numbers.
 - (a) $z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$
 - (b) $\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 \theta_2) + i\sin(\theta_1 \theta_2)], \text{ where } r_2 \neq 0.$
 - (c) $[r(\cos\theta + i\sin\theta)]^n = r^n [\cos(n\theta) + i\sin(n\theta)]$
 - (d) $[r(\cos\theta + i\sin\theta)]^{\frac{1}{n}} = \sqrt[n]{r} [\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}],$ where $k = 0, 1, \dots, n-1.$

34.2 Write the trigonometric forms of the complex numbers.

- (a) $1 + \sqrt{3}i$
- (b) 1+i
- (c) *i*
- (d) 1
- (e) $\frac{1}{i}$
- **34.3** Compute square roots: $\sqrt{-i}$.
- **34.4** Solve for complex values of x in $x^5 = 1$.
- **34.5** If complex number z is an 8th root of unity, is $\sqrt[8]{6z}$ an 8th root of 6?
- **34.6** Suppose a cube root of a complex number z is $2(\cos a + i \sin a)$. Express the two other cube roots of z in terms of a.
- **34.7** Suppose two *n*-th roots of complex number z are $\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$ and $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$. Find values of 10 more roots. Do not express them in terms of n.

```
© Qishen Huang
```

- **34.8** Compute $(\sqrt{3}+i)^8$.
- **34.9** Define operations on ordered pairs of real numbers like (x, y) as follows:

$$(u, v) \pm (x, y) = (u \pm x, v \pm y).$$

 $k(u, v) = (ku, kv), \text{ where } k \text{ is a real constant}$
 $(u, v) \times (x, y) = (ux - vy, uy + vx).$

Verify two identities.

- (a) (u, v) + (x, y) = (x, y) + (u, v)
- (b) $(u, v) \times (x, y) = (x, y) \times (u, v)$
- 34.10 Use the definition in Problem 34.9. Solve equation

$$(x, y) \times (x, y) = (-1, 0).$$

34.11 Use the definition in Problem 34.9. Solve equation

$$(x, y) \times (x, y) \times (x, y) = (1, 0).$$

34.12 Use the definition in Problem 34.9. For pair (u, v) with $uv \neq 0$, we define its reciprocal as pair (x, y) such that

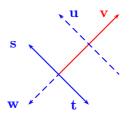
$$(u, v) \times (x, y) = (1, 0).$$

Derive a formula for reciprocal.

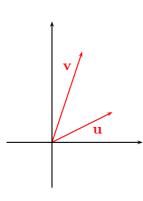
- **34.13** We define an operation for number pairs like (r, θ) , where $r \ge 0$ and $\theta \in R$, as follows:
 - $(r_1, \theta_1) \times (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2).$
 - (a) Show that $(r_1, \theta_1) \times (r_2, \theta_2) = (r_2, \theta_2) \times (r_1, \theta_1)$.
 - (b) Solve for x and y in equation $(r, \theta) \times (x, y) = (1, 0)$, where $r \neq 0$.
- **34.14** We continue to use the definition of multiplication in the previous problem. We add a new definition: (r_1, θ_1) is said to be equal to (r_2, θ_2) if $r_1 = r_2$ and $\theta_1 - \theta_2$ is a multiple of 2π . Solve equation $(r, \theta) \times (r, \theta) = (1, 0)$, where $r \ge 0$ and $\theta \in [0, 2\pi)$.

Vectors and Matrices

35.1 Vector **v** is given on the graph below. Find the vector of $-\frac{1}{2}$ **v** on the graph.



35.2 Given vectors \mathbf{u} and \mathbf{v} , draw vectors $\mathbf{u} - \mathbf{v}$ and $\frac{1}{2}(\mathbf{u} + \mathbf{v})$.



- **35.3** Express vector $\mathbf{v} = (2, 3)$ in terms of two unit vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.
- **35.4** Vectors $\mathbf{u} = (a, b)$ and $\mathbf{v} = (c, d)$. Prove that

$$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|.$$

35.5 Identify true statements.

 \bigodot Qishen Huang

- (a) If vector \mathbf{u} is perpendicular to vector \mathbf{v} , then inner product $\mathbf{u} \cdot \mathbf{v}$ is zero.
- (b) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
- (c) If $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\|$, then component of \mathbf{u} along \mathbf{v} is $\|\mathbf{u}\|$.
- (d) $||k\mathbf{u}|| = k||\mathbf{u}||$ for any constant k.
- **35.6** Matrices A and B have orders of $n \times n$. Matrix I is identity matrix. Identify true statements.
 - (a) AB = BA
 - (b) $I_{n \times n} A = A$
 - (c) $|I_{n \times n}| = 1$ for all $n \ge 1$.
 - (d) |A + B| = |A| + |B|
 - (e) |kA| = k|A|, where k is a constant.
 - (f) If A has two identical rows, then |A| = 0.
 - (g) If |A| = 0, then A is not invertible.
- **35.7** Square matrix A of order 2 has determinant of 1. Find the determinant of matrix 3A.
- **35.8** Matrices A and B of order 2×2 satisfy AB = I. Compute the product BA.
- **35.9** Matrix A has order of $m \times n$. Find the order of product AA'. Which elements in the product are definitely not negative?
- **35.10** Matrices A and B are of order 4×4 .

$$B = \left(\begin{array}{rrrrr} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

Which rows or columns of A - BA may have non-zero elements?

- **35.11** Matrix A is invertible. Find the inverse matrix of A^{-1} .
- 35.12 Suppose invertible matrices A and B are of same order. Identify true statements.

(a)
$$|A^{-1}| = |A|^{-1}$$

(b) $(AB)^{-1} = A^{-1}B^{-1}$

35.13 Square matrix A has determinant of 2. Find the value of determinant of its inverse matrix A^{-1} .

```
© Qishen Huang
```

- **35.14** Two rows of a 2×2 matrix are exchanged. Compare the determinants before and after the exchange.
- **35.15** Add the first row to the second row in a 2×2 matrix. Compare the determinants before and after the addition.
- **35.16** Two rows of a 2×2 matrix are identical. Compute the determinant of the matrix.
- **35.17** Matrices A and B have orders of 1×2 and 2×1 . Compute determinant |AB|.

Parameterized Equations

36.1 Are the curves defined by parameterized equations identical?

- (a) $x = \sin(t^2), \quad y = \cos(t^2), \quad t \in R.$
- (b) $x = \sin t$, $y = \cos t$, $t \in R$..
- 36.2 Are the curves defined by parameterized equations identical?
 - (a) $x = \sin t$, $y = \cos t$.
 - (b) $x = -\sin t$, $y = \cos t$.

36.3 Are the curves defined by parameterized equations identical?

(a)
$$x = f(t), \quad y = g(t), \quad t \in R.$$

(b) $x = f(t-1), \quad y = g(t-1), \quad t \in R.$

36.4 Find the curve defined by parameterized equations

$$x = -\frac{1}{t}$$
, $y = t + 1$, where $t \neq 0$.

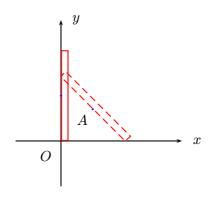
36.5 Find the curve defined by parameterized equations

$$x = -t$$
, $y = t^2 + 1$, where $t \in R$.

36.6 Find the curve defined by parameterized equations

$$x = -\sin t$$
, $y = 2\cos t + 1$, where $t \in [0, 2\pi)$.

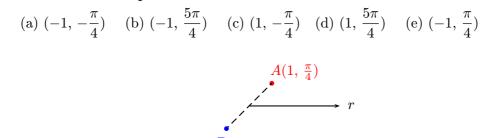
36.7 A ruler by the wall falls down. The top end lands at the origin point. Write parameterized equations of the midpoint of the ruler. Is it a line segment?



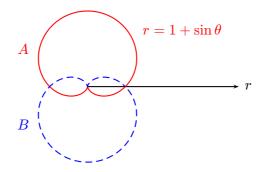
Polar Coordinates

37.1 In which field below do you think polar equations are most useful?

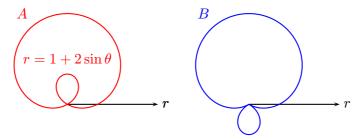
- (a) Astronomy
- (b) Business
- (c) Chemistry
- (d) Economics
- **37.2** On the $r\theta$ -plane, can the coordinates be negative?
- **37.3** Find the point symmetric to (r, θ) with respect to the polar axis.
- **37.4** Given point $A(1, \frac{\pi}{4})$ on the polar coordinate plane, identify the coordinates of B.



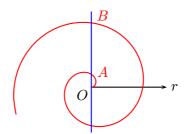
37.5 Curve A represents polar equation $r = 1 + \sin \theta$, where $\theta \in [0, 2\pi)$. Find the polar equation of curve B.



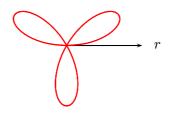
- **37.6** Rotate $r = 1 \sin \theta$, where $\theta \in [0, 2\pi)$, clockwise around the pole by 30°. Write the new equation.
- **37.7** Curve A represents polar equation $r = 1+2\sin\theta$, where $\theta \in [0, 2\pi)$. Find the equation of curve B.



37.8 The curve represents polar equation $r = \theta$. The vertical line is perpendicular to the polar axis. Find the length of line segment AB.



37.9 Given the graph of $r = \sin(k\theta)$ where $\theta \in [0, 2\pi)$, find integer value of k. Identify the portion of the function on $[\frac{3}{2}\pi, 2\pi)$.



Statistics

- **38.1** Identify the purposes of these statistics: mean, mode, median, quartile, and standard deviation.
 - (a) Measure of central tendency
 - (b) Measure of dispersion (or variation)
- **38.2** To measure central tendency in nominal (categorical) data, which statistic is usually best: mean, median, or mode?
- 38.3 Would an outlier of a sample affect more the range or quartiles of the sample?
- **38.4** A sample has outliers at the extremes of the data set. To measure central tendency, which statistic is better: mean or median?
- **38.5** A sample has either ratios or interval data that are most skewed. To measure central tendency, which statistic is usually better: mean or median?
- **38.6** What is the best reason not to use a stem plot?
 - (a) Too much data
 - (b) Outliers
 - (c) Skewed data
 - (d) Not bell shaped data
- **38.7** If an element is removed from a sample, which statistics of the sample may not be affected?
 - (a) Mean (b) Mode if exists (c) Median
- 38.8 Identify true statements about a sample.

```
© Qishen Huang
```

- (a) A sample always has a finite number of elements.
- (b) A sample is usually a subset of the whole population.
- (c) A larger sample always represents more accurately the whole population.
- (d) Sample variance is always positive.

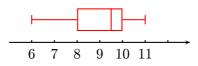
38.9 Identify true statements about a sample.

- (a) Mode may not exist.
- (b) Some element is equal to the mean.
- (c) Some element is equal to the mode, if the mode exists.
- (d) Some element is equal to the median.
- (e) Mean and average are same.
- (f) The second quartile is the median.
- (g) The range of a sample is a closed interval with minimum at left end and maximum at the right end.
- **38.10** The median of a sample may not be same as the mean. Give an example of each scenario.
 - (a) Median is equal to mean.
 - (b) Median is greater than mean.
 - (c) Median is less than mean.
- **38.11** Identify an outlier in sample $\{1, 2, 3, 4, 5, 10\}$.
- **38.12** Choose a pie chart or a bar chart to use in each scenario.
 - (a) Percentages of grades of a math class.
 - (b) Numbers of students by age in a high school.
- **38.13** Consider a stem plot and a histogram for the same data set:

44, 46, 47, 49, 63, 64, 66, 68, 68, 72, 72, 75, 76, 81, 84, 88, 106.

The stem plot has leaf unit of 1 and stem unit of 10. The histogram has interval width of 10. Which plot presents more information?

38.14 Reading the box plot below. Find the minimum, maximum, and median. Is the sample data left or right skewed?



38.15 Which statistic of sample $\{x_i\}$ more likely provides a more accurate estimate of its variance:

$$\frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n} \quad \text{or} \quad \frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n - 1}?$$

- **38.16** Sample $A = \{x_1, x_2, ..., x_{10}\}$. Find the value of *a* that minimizes $\sum_{i=1}^{10} (x_i a)^2$.
- **38.17** Combine two samples to form a new sample. Can the variance of the new sample be less than the two old variances?
- **38.18** A sample of 20 observations has mean of 10. Later, 4 more observations are added: 12, 13, 14, and 15. Find the new mean.
- **38.19** Samples A and B have means of 3 and 4, respectively. Sample $C = A \cup B$. Which is true about the mean of C?

(a) = 3.5 (b) ≤ 3 (c) ≥ 4 (d) $\in (3, 4)$

- **38.20** Samples A and B have means of 10 and 15, respectively. Drop half elements from each sample. Is the new mean of sample A less than that of sample B?
- **38.21** Order is important in permutation and not in combination. In which case below is order considered important?
 - (a) A line of people in front of cashier to a customer.
 - (b) Test scores of a class to the teacher.
- **38.22** Identity true statements.
 - (a) If an event has probability of 1, it always occurs.
 - (b) If an event has probability of 0, it never occurs.
 - (c) If an event always occurs, it has probability of 1.
 - (d) If an event never occurs, it has probability of 0.
 - (e) If an event has probability of 0.5, it occurs exactly 50% of time.
- **38.23** The chance of winning a lottery is exactly 0.1%. If John buy 10 different tickets, what is his chance of winning?

[©] Qishen Huang

- **38.24** In $\triangle ABC$, point P is a random point on side BC. What is the chance that the area of $\triangle ABP$ is more than one third of that of $\triangle ABC$?
- **38.25** Five highest scoring students A, B, C, D, and E have scores of 81, 82, 83, 84, and 85. In the award ceremony, they stand in a line. The highest scorer in the middle. The next two highest on his two sides. The remaining two at the ends. How many possible ways are there to place them?

Limits

39.1 Suppose $\lim_{n\to\infty} a_n = 1$. Which statements are true?

- (a) All terms may not be 1.
- (b) All terms after some term are between 1 0.0001 and 1 + 0.0001.
- (c) The sequence is bounded.
- (d) Add, remove, or change a few terms. The new sequence still has the same limit.
- (e) Remove these terms $\{a_{2n}, n = 1, 2, 3, ...\}$. The new sequence still has the same limit.
- (f) Add 1 to each of these terms $\{a_{2n}, n = 1, 2, 3, ...\}$. The new sequence $\{a_n\}$ still has some limit.
- (g) The limit of sequence $\{a_n a_{n-5}, n > 5\}$ is zero.

39.2 Suppose $\lim_{n\to\infty} a_n = \infty$. Which statements are true?

- (a) After a certain term, all other terms after the term are positive.
- (b) Any subsequence goes to ∞ as n goes to ∞ .
- (c) After a certain term, the sequence $\{a_n\}$ is increasing.
- (d) The limit of sequence $\{a_n a_{n-5}\}$ is zero.

39.3 Suppose $\lim_{n \to \infty} a_n = 2$. Compute following limits.

- (a) $\lim_{n \to \infty} (a_n 3)$
- (b) $\lim_{n \to \infty} |a_n|$
- (c) $\lim_{n \to \infty} a_n^2$
- (d) $\lim_{n \to \infty} e^{a_n}$

- (e) $\lim_{n \to \infty} 3a_n$
- (f) $\lim_{n \to \infty} a_{2n}$
- (g) $\lim_{n \to \infty} a_{n^2}$

39.4 Suppose $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Compute the following limits.

- (A) $\lim_{n \to \infty} (a_n + b_n)$
- (B) $\lim_{n \to \infty} (a_n b_n)$
- **39.5** Suppose $\lim_{n \to \infty} a_n = 2$ and $\lim_{n \to \infty} b_n = 3$. In addition, $b_n > 0$ for all $n \ge 1$. Compute the following limits.
 - (a) $\lim_{n \to \infty} \frac{a_n}{b_n}$ (b) $\lim_{n \to \infty} \log b_n$ (c) $\lim_{n \to \infty} \sqrt{b_n}$
- **39.6** If sequence $\{a_n\}$ is increasing, then $\lim_{n \to \infty} a_n$ either exists or is ∞ . Give an example for each scenario.
- **39.7** Provide an example of $\{a_n\}$ satisfying both requirements:

39.8 Suppose $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 1$ and $a_n \le c_n \le b_n$, where $n \ge 1$. Proved $\lim_{n \to \infty} c_n = 1$.

- **39.9** If $\lim_{n \to \infty} (a_n b_n)$ exists, does $\lim_{n \to \infty} a_n$ or $\lim_{n \to \infty} b_n$ exist?
- **39.10** If $\lim_{n\to\infty} (a_n b_n)$ and $\lim_{n\to\infty} a_n$ exist, does $\lim_{n\to\infty} b_n$ exist?
- **39.11** Suppose $\lim_{n \to \infty} (a_n b_n) = 1$ and $\lim_{n \to \infty} a_n = 2$. In addition, $a_n \neq 0$ for $n \ge 1$. Compute $\lim_{n \to \infty} b_n$.
- **39.12** Define sequence $\{a_i\}$ as follows:

 $a_1 = 0, a_n = \sqrt{a_{n-1} + 1}$, where n > 1.

It is known $\lim_{n \to \infty} a_n$ exists. Find the limit.

39.13 Function $f(x) = x^2$. Define sequence $\{a_n, n = 1, 2, 3, ...\}$ as follows:

$$a_n = \frac{f(1+\frac{1}{n}) - f(1)}{\frac{1}{n}}.$$

Compute $\lim_{n \to \infty} a_n$ if exists.

39.14 Computer the limit if exists:

$$\lim_{n \to \infty} (\sqrt{n+1} - \sqrt{n}).$$

39.15 Function $g(x) = \sqrt{x}$. Define sequence $\{a_n, n = 1, 2, ...\}$ as follows:

$$a_n = \frac{g(1-\frac{1}{n}) - g(1)}{-\frac{1}{n}}.$$

Compute $\lim_{n \to \infty} a_n$ if exists.

39.16 Compute the limit if exists:

$$\lim_{n \to \infty} [\sin(x + \frac{1}{n}) - \sin x].$$

39.17 The domain of function f is R. In addition, $\lim_{x\to\infty} f(x) = 2$. Identify true statements.

- (a) $\lim_{n\to\infty} f(n) = 2$, where n is integer.
- (b) If $\lim_{n \to \infty} a_n = \infty$, then $\lim_{n \to \infty} f(a_n) = 2$. (c) $\lim_{x \to \infty} f(e^x) = 2$.
- (d) If $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} f(g(x)) = 2$.
- **39.18** The domain of function f is R. $\lim_{n\to\infty} f(n) = 1$, where n is an integer. Does it imply $\lim_{x\to\infty} f(x) = 1$?

39.19 Find the limits of functions at infinity, if exist.

(a)
$$2x + 3$$
.
(b) $\frac{1}{2x+3}$, where $x > 0$.
(c) $\frac{2x+3}{x+1}$, where $x > 0$.
(d) $\frac{x^2 - 1}{x+1}$, where $x > 0$.

- (e) $x^5 2x^4$.
- (f) \sqrt{x} , where x > 0.
- (g) e^x .
- (h) $\ln x$, where x > 0.
- (i) $\sin x$.
- (j) $\tan^{-1} x$.

39.20 Identify true statements.

- (a) That $\lim_{x \to x_0} f(x)$ exists does not mean x_0 is in the domain of the function.
- (b) If x_0 is in the domain of f and $\lim_{x \to x_0} f(x)$ exists, then $\lim_{x \to x_0} f(x) = f(x_0)$.
- (c) If $\lim_{x \to x_0} f(x)$ exists, then $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ exist and are equal.
- (d) If for any sequence $\{a_n\}$ with limit of 0 and $a_n \neq 0$ for all $n \ge 1$, $\lim_{n \to \infty} f(x_0 + a_n)$ exists and the limit does not depend on $\{a_n\}$, then $\lim_{x \to x_0} f(x)$ exists.
- **39.21** Function f(x) = [x], the largest integer not more than x. Does f(x) have a limit at x = 8?
- **39.22** If $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 2$ and $\lim_{x \to x_0} g(x) = 3$, compute $\lim_{x \to x_0} f(x)$.
- **39.23** If $\lim_{x \to x_0} f(x) = -1$ and $\lim_{x \to x_0} g(x) = \infty$, compute $\lim_{x \to x_0} (f(x)g(x))$.
- **39.24** Compute the limit if exists:

$$\lim_{x \to 0} \frac{x}{x^2 + 1}$$

- **39.25** Jack and Jim play a number game. Jack writes a number 1.5 secretly. Jim also writes a real number x. If x = 1.5, Jack pays Jim \$10. Otherwise, Jim receives nothing.
 - (a) Write the function of the amount of money Jim receives.
 - (b) Show that the function has a limit at 1.5.
 - (c) Is the limit equal to f(1.5)?