Washington State Mathematics Standards

Review and Recommendations

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Better learning by design

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Statement of Intent

Washington's State Board of Education requested an independent review to analyze the strengths and weaknesses of Washington's current K-12 mathematics standards. The ultimate goal is to ensure the K–12 education Washington students are receiving prepares them to successfully enter the world of work and postsecondary training with the applied skills, computational fluency and conceptual knowledge they need.

This set of recommendations is intended to guide Washington State's Office of the Superintendent of Public Instruction (OSPI) as it revises the current mathematics standards. This is the first step in the systematic review of mathematics education in Washington State, as outlined in the Joint Mathematics Action Plan.[1]

To aid OSPI, we reviewed and benchmarked the Essential Academic Learning Requirements (EALRs) and the Grade Level Expectations (GLEs) in Washington's mathematics standards to exemplary states, countries and national framework standards. OSPI has been given the review reports and the data generated by the benchmarking process to use as it rewrites Washington's standards.

The review does not include an examination of all of the documents that define and support the standards nor does it include a review of Washington's Assessment of Learning and its supporting documents. Also excluded are the examples under the GLEs, because they are specifically described by OSPI as "examples of how students may show mastery of the expectations. They are not a checklist and in no way should anyone believe that teachers need to teach to all of these.[2]"

We at Strategic Teaching recognize that standards are only one piece of a complex, interconnected system, and in terms of what is expected of mathematics education, a review of just the

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[1] State of Washington, State Board of Education, *Joint Mathematics Action Plan: Building the Proper Foundation*, November 30, 2006, retrieved June 16, 2007, from www.sbe.wa.gov/gradreq/caa/docs/JointMathematicsActionPlan11-30-06.pdf

[2] Letter from Dr. Terry Bergeson, Superintendent of Public Instruction, September 8, 2006, retrieved May 12, 2007, from http://www.k12.wa.us/assessment/WASL/Mathematics/MathGLEDraft.aspx

standards is, by definition, incomplete. Alone, standards do not improve teaching and learning.

However, standards must come first. They are the official statement of what students should be taught. Changes in the standards will set in motion changes to the entire system, including aligned curriculum, instruction, assessment, and professional development.

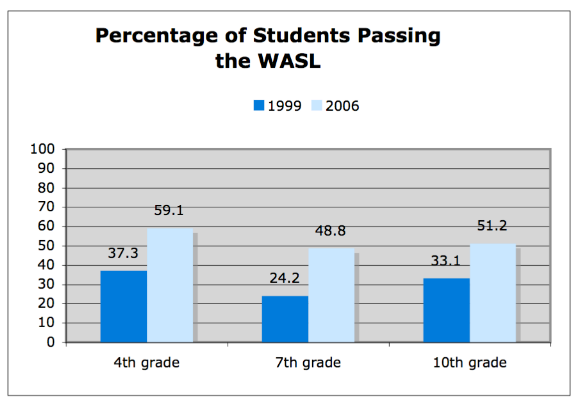
Executive Summary

At the request of the State Board of Education, Strategic Teaching spent the spring of 2007 examining Washington's mathematics standards. Using teams of accomplished reviewers, we compared the Washington standards to those in several key states, as well as to standards published by national groups and high performing countries.

Using a set of nine rubrics (Appendix A) Strategic Teaching evaluated the content, rigor, specificity, clarity, depth, grade-to-grade coherence, measurability, accessibility, and balance in Washington State Mathematics Standards. We evaluated these characteristics using a mix of item-by-item comparisons and a global review of the standards document.

The bottom line is that Washington's mathematics standards need to be strengthened. If mathematics is the gateway to student success in higher education and the workplace, Washington is getting too few of its students to and through the door.

To be sure, there are good qualities in Washington's standards, including the well-defined and developed mathematical processes related to defining and solving problems, reasoning, connecting mathematical concepts to each other and to the larger world, and communicating about mathematics. There are also some well-developed strands, such as early elementary algebra.



The hard work of educators across the state is reflected in the state's relatively high scores on the National Assessment of Educational Progress, the SAT, and the ACT. And Washington is moving in the right direction as measured by the Washington Assessment of Student Learning (WASL). The number of students successful on the WASL in mathematics has increased to the point that 62 percent of the students who took the test in June 2007 have now passed.

However, compared to the standards of key states and high achieving countries, Washington is not expecting enough of its students. There is insufficient emphasis on core mathematical content. Some math should be taught earlier in a student's schooling, and some crucial math is missing completely.

Simply put, Washington is not focused enough on the important fundamental content topics in mathematics.

This is shown in the early grades in which Washington standards do not ensure that students learn the critical algorithms of arithmetic and continues throughout the standards until it ends in secondary school with minimal expectations that are missing most of the algebra, geometry, and trigonometry found in other places.

Washington does not provide sufficient clarity in its mathematics standards about what and *how well* mathematics should be learned or adequate guidance about what students should be able to do with the mathematics after it is learned. For example, the standards often call for student "understanding" rather than specifying how a student can demonstrate understanding through problem solving, estimation, or calculation.

In this report, we present seven recommendations that, if implemented, will bring to the surface the most important mathematical content, provide greater clarity about what is expected of students in each grade, provide more explicit guidance to educators about what to teach when, and ultimately result in more Washington students having the opportunity to succeed in mathematics.

Our recommendations, which are organized with the most important at the top of the list, are:

1. Set higher expectations for Washington's students by fortifying content and increasing rigor.
2. Make clear the importance of all aspects of mathematics: mathematics content, including the standard algorithms; the conceptual understanding of the content; and the application of mathematical processes[3] within the content.
3. Identify those topics that should be taught for extended periods at each grade level and better show how topics develop over grade levels.

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[3] Washington State defines the following process strands: communicating, reasoning, problem solving, and making connections with mathematical content.

4. Increase the clarity, specificity, and measurability of the Grade Level Expectations.
5. Write Essential Academic Learning Requirements that restructure the standards document to clarify grade-level content priorities and that reflect both the conceptual and procedural sides of mathematics.
6. Create a standards document that is easily used by most people.
7. Create small, expert Standards Revision Teams for each grade band and systematically collect feedback on the revised standards.

Process

A set of nine rubrics, on a 1 – 4 point scale, is used to compare and judge the EALRs and GLEs. Appendix B contains summaries of the rubrics. The rubrics, written to examine traits identified by the State Board of Education, were designed to answer the following questions:

- Content: Does Washington include the same mathematical content as other, well-respected standards documents?
- Rigor: Is the content present at the same grade levels? Are students expected to apply that content in demanding ways?
- Specificity: Are the GLEs written with the same amount of detail as other documents?
- Clarity: Is it easy to understand what the GLEs mean?
- Depth: Are important mathematics topics fully developed?
- Grade-to-grade coherence: Do topics develop logically and sequentially over grade levels?
- Measurability: Can the GLEs be assessed?
- Accessibility: Are Washington's standards easy to use for as many people as possible?
- Balance: Is it clear that mathematical content (including algorithms), conceptual understanding, and mathematical processes are present in Washington's standards?

The documents used in this review are (1) California State Standards, (2) Massachusetts State Standards, (3) Indiana State Standards, (4) Singapore Curriculum, (5) Finland Standards, (6) *Curriculum Focal Points,* (7) the National Assessment for Education Progress (NAEP), (8) the American Diploma Project, and (9) the Washington College Readiness Math Standards.

These documents were selected carefully. The three states were chosen because they are the only three states whose standards received the highest rating from the three organizations currently reviewing state standards: American Federation of Teachers, the Fordham Foundation, and Editorial Projects in Education. Finland and Singapore were chosen because they are, respectively, the top-scoring countries on the Program for International Student Assessment and the Trends in International Mathematics and Science Study. *Curriculum Focal Points* was selected because it was developed and has been well received by people with varying views about teaching mathematics. The National Assessment for Education Progress was used because Washington's students take this assessment, and the results are valued. The Washington College Math Readiness Standards were selected because they are expected to have an impact on students transitioning to college in Washington. And last, the American Diploma Project's Benchmarks were chosen because Washington State has recently joined a coalition with 29 other states to support these standards. Appendix A details the documents used for comparisons for each of the examined grade levels.

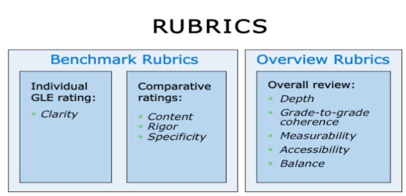
None of these (or any other) standards documents are perfect, but each has characteristics and content that can inform the work of revising Washington's standards.

As previously mentioned, having good standards does not guarantee high student achievement. Education systems are much more complex than that. Good standards are necessary but insufficient.

Content, rigor, and specificity are judged by comparing each standard in the above-mentioned documents to their matching GLE and then rating the degree of match, based on a set of rubrics, scored from 1 to 4, with 4 being high. The rubrics used throughout the examination of Washington's standards were specifically created for this project.

Clarity is judged by examining each GLE independent of other documents and rating it between 1 and 4, again using a rubric.

The last five characteristics were evaluated in a separate review from a global perspective after a thorough examination of Washington State Mathematics Standards.



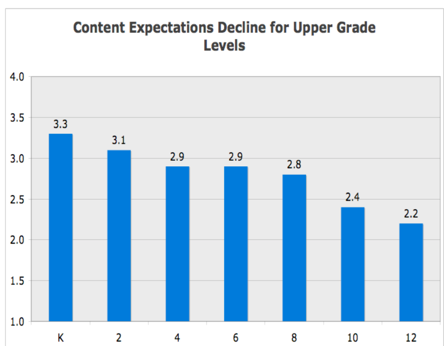
This chart illustrates the three approaches used to evaluate Washington's standards.

More than 21,000 comparisons between standards in exemplar documents and GLEs were made during the process. Using the scores from these comparisons, we found the mean (i.e. the average) and a weighted mean that emphasized low scores. The weighted mean was designed to show, for example, how often content was missing completely in Washington. For our charts, we chose to use the usual mean for simplicity.

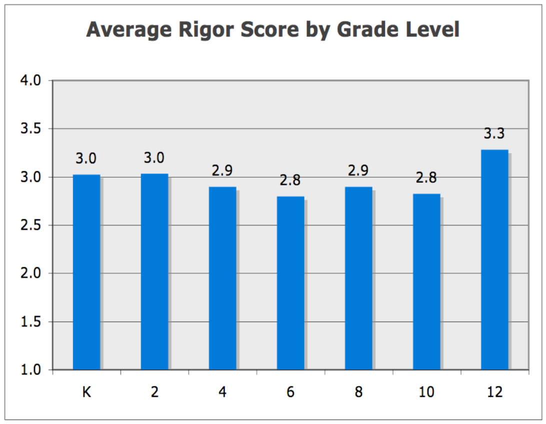
Six reviewers (Appendix C), working in teams of two, completed the reviews and then worked to come to a consensus on the score points. The reviewers were chosen because of their knowledge of mathematics, their grade-level expertise, and their experience with standards.

Findings

After using the 1-to-4 point rubrics to find and rate all of the matching standards, we identified the mean of the scores of all of the documents examined at that grade level. We then found the average of all of the averages for the documents at each grade level. The following charts show these means — averages of averages — for the traits of content, rigor, and specificity.

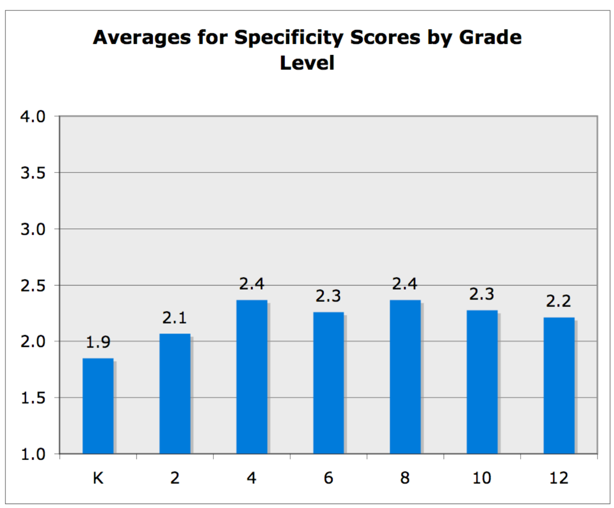


Today, Washington expects its students to learn less content than do other states and high-achieving countries. During the elementary years, Washington earns an average score of about 3 on a 1-4 scale. This means Washington's students see a little more than two-thirds or about 68 percent of the content in the other standards documents that were reviewed. By grade 12, that drops to 40 percent. (On a 1-4 point scale, 1 is 0%, 2 is 33%, 3 is 67% and 4 is 100%.)

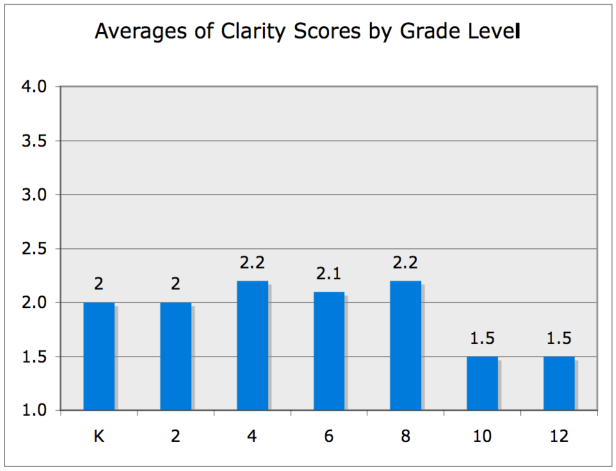


Washington's rigor score hovers around 3 across all grade levels when compared to the other examined documents. Generally, this means that when similar mathematics content is present at the same grade levels, Washington's students are expected to do less demanding work with that content.

It's important to note that rigor is scored only when there is a content match. It is impossible to make a judgment about how rigorous Washington is compared to another document when Washington is missing the content.



Washington's GLEs are a lot less specific than standards in other documents. Sometimes Washington has less detail about the content, sometimes there is less detail about what students should do with the content, and sometimes both areas are vague.



Clarity scores are low, ranging from 1.5 to 2.2. The score of 2 is given when the GLE is unclear, but the examples under the GLE help. A score of 1 is given when, even after reading the examples, we do not know what the GLE means.

The lack of clarity stems partly from GLEs that are written at the general, rather than specific, level. Another cause is Washington's repeated use of the verb "understand," which is open to multiple interpretations.

The following examples illustrate strong matches, 4s, between a GLE and its matching standard from another document. Note that a 4 score on one trait does not mean that the match would score 4 on other traits.

These are the kind of matches we wanted to see more often.

Content: This example shows a good match between Washington and Massachusetts.

WA grade 8, GLE 1.4.1	**Massachusetts grade 8, 8.D.4**
Understand the concept of compound events.	Use tree diagrams, tables, organized lists, basic combinatorics ("fundamental counting principle"), and area models to compute probabilities for simple compound events, e.g., multiple coin tosses or rolls of dice.

Washington earns a 4 for matching the content in the Massachusetts standard because both standards are about the same mathematics topic: compound events. Rigor scores a 3 (the content in both documents is at the same grade level, but Washington expects its students to do less demanding work with that content); specificity scores a 2 (Massachusetts is much more detailed); and clarity scores a 2 (the GLE is vague, but the examples clarify the meaning).

Rigor: The following example shows a good match between Washington and California.

WA grade 4, GLE 1.5.4	California grade 4, A.1.1
Use a single variable to write expressions and equations that represent situations involving multiplication and division of whole numbers.	Use letters, boxes, or other symbols to stand for any number in simple expressions or equations (e.g., demonstrate an understanding and the use of the concept of a variable).

Washington earns a 4 for matching the rigor in the California standard because the content is at the same grade level in both documents and Washington has similar expectations about what students should do with the content. Content also scores a 4 (although the language doesn't match, these two standards are talking about the same content); specificity scores a 3 (California is slightly more detailed); and clarity scores a 3 (this GLE is fairly clear, but there is some question about what "situations" means).

Specificity: The following example shows a good match between Washington and Singapore.

WA grade 8, GLE 1.3.3	Singapore One: Functions and graphs
Describe the relative position of points on a coordinate grid.	▪ Cartesian coordinates in two dimensions ▪ Graph of a set of ordered numbers

Washington earns a 4 for matching the specificity in the Singapore standard; the two standards are written at about the same level of detail. Content also scores a 4 (the mathematical topics are the same); rigor is not rated (Singapore lacks verbs, so it is impossible to compare the expectations of the two standards); and clarity is rated 3 (what is meant by "relative position" is not completely clear).

Next, we share examples that show weak matches between the GLEs and standards from the other documents.

Content: The following example shows a 1 match between Washington and Indiana.

WA, grades 9 and 10	**Indiana, Integrated Math I and II**
	▪ Statistics: – Find skewness and symmetry from a graph of data. – Find a linear transformation. Example: Consider the following data: 6, 4, 4, 6, 8, 10, 2, 5, 9. Suppose that 5 is added to each value. Compare the mean and average mean deviation of the original and new data sets. – Use the Law of Large Numbers to understand situations involving chance. Example: A class is flipping coins to study probability. In one experiment, the coin was flipped 1,000 times. The next day, the coin was flipped 2,000 times. Which experiment — the 1,000 flips or the 2,000 flips — has the highest probability of getting 50 percent heads? Why?

Indiana has 36 standards that have no match in Washington's standards. The standards related to statistics are shown above. Additionally, Indiana includes standards about proving theorems, advanced geometry, graph theory, matrices, logic, work with circles and spheres, and the quadratic equation. Washington was given a 1 for each of the topics missing in Washington and found in Indiana. Since there is no content match, it is not possible to score for other attributes, and those scores are left blank.

Specificity: The following shows a 1 match between Washington and Indiana.

WA grade 9 and 10, GLE 1.5.6	IN, Integrated Math I
Apply properties to solve multi-step equations and systems of equations.	▪ Solve linear equations. Example: Solve the equation 7a + 2 = 5a - 3a + 8. ▪ Solve equations and formulas for a specified variable. Example: Solve the equation q = 4p - 11 for p. ▪ Find solution sets of linear inequalities when possible numbers are given for the variable. Example: Solve the inequality 6x - 3 > 10 for x in the set {0, 1, 2, 3, 4}. ▪ Solve linear inequalities using properties of order. Example: Solve the inequality 8x - 7 ≤ 2x + 5, explaining each step in your solution. ▪ Use a graph to estimate the solution of a pair of linear equations in two variables. Example: Graph the equations 3y – x = 0 and 2x + 4y = 15 to find where the lines intersect. ▪ Understand and use the substitution method to solve a pair of linear equations in two variables. Example: Solve the equations y = 2x and 2x + 3y = 12 by substitution. ▪ Understand and use the addition or subtraction method to solve a pair of linear equations in two variables. Example: Use subtraction to solve the equations: 3x + 4y = 11, 3x + 2y = 7. ▪ Understand and use multiplication with the addition or subtraction method to solve a pair of linear equations in two variables. Example: Use multiplication with the subtraction method to solve the equations: x + 4y = 16, 3x + 2y= -3.

Washington earns a 1 for specificity because Washington is much less specific than Indiana. Washington earns a 3 for content (the GLE and the Indiana standards relate to the same content, and the case can be made that the GLE includes the Indiana content); rigor scores a 3 (the content is at the same grade level but is less demanding); and clarity scores a 2 (the examples add the clarity that is missing in the GLE).

Rigor: The following is an example of a 1 match between Washington and California.

WA grade 8, GLE 1.4.2	**CA, grade 6**
Use procedures to compute the probability of dependent and independent events.	▪ Compute the range, mean, median, and mode of data sets. ▪ Understand how additional data added to data sets may affect these computations of measures of central tendency. ▪ Understand how the inclusion or exclusion of outliers affects measures of central tendency. ▪ Identify data that represent sampling errors and explain why the sample (and the display) might be biased. ▪ Identify claims based on statistical data and, in simple cases, evaluate the validity of the claims.

Washington earns a 1 for rigor because Washington's GLE is at a later grade level and it requires less demanding work of the student. Washington earns a 2 for content (the content is present in Washington, but it is less sophisticated); specificity scores 1 (Washington is much less detailed than California); and clarity scores 2 (the examples clarify the term "procedures").

A second, independent, review of the Washington State Mathematics Standards document as a whole was done for the traits of depth, grade-to-grade coherence, measurability, accessibility, and balance. The overall review does not rely on any item-by-item but rather on a global review of the standards document.

The different methodology and the examination of different traits produced scores that are much lower than the scores from the benchmarking process. One reason for this is because there can be a partial match of content, for example, even if the most important aspects of content are missing.

Since one of the review team served in both roles, reviewer and benchmarker, we are confident in the results, despite the apparent disconnect between the scores.

Scores from the overall review

Depth	Washington earns a 1 because numerous topics are missing and many of the topics that are included are underdeveloped.
Grade-to-grade coherence	Washington earns a 1 because there are serious interruptions in the content, and single topics are sometimes scattered throughout the document, making it difficult to determine how a topic develops over grade levels.
Measurability	Washington earns a 1 because the content is often vaguely defined and because verbs are unobservable and therefore not measurable.
Accessibility	Washington earns a 1 because the format makes the standards difficult for most people to understand and easily use.
Balance	Washington earns a 1 because the document underemphasizes mathematical content and algorithms.

The central problem is that crucial core content, such as fluency with the standard arithmetic algorithms, algebra II, and most of geometry, is missing. Without the core content, the traits of grade-to-grade coherence, measurability, accessibility, and balance also are missing.

Areas of Strength

Certainly there are positive aspects of Washington's current standards that deserve mention.

- Conceptual understanding and mathematical processes are well presented in the standards.
- The grade-to-grade connections are stronger at the elementary grades.
- The side-by-side presentation format of the GLEs shows well the interconnectedness and maturation of the mathematics in Washington's standards. This format supports the vertical articulation that is essential to good mathematical instruction and should continue to be offered in addition to other formats discussed later.
- The early elementary algebra strand is strong and does a good job of grade articulation.

General Areas of Concern

The reviewers identified the following areas as the most problematic:

- Understanding mathematics outweighs concrete mathematics. Washington's standards do not adequately spell out the concrete mathematics that is to be understood. Without specific content, it is not possible to be precise about what is meant by "understanding."
- Algorithms are missing or sidestepped in Washington's elementary school standards. From a teaching support perspective, the area most in need of development is a thorough and connected set of computation algorithms across all strands and all elementary grades.
- EALRs are poorly organized. Each EALR has multiple components; each component has subcomponents; each subcomponent has multiple GLEs, which are further organized by grade level; and most GLEs have multiple examples. The five layers are unnecessary and confusing. Appendix D shows samples of how other state standards documents are organized.
- GLEs are not articulated by high school mathematics course. From an outsider's perspective, Washington's standards do not seem helpful to secondary teachers.
- Verbs, which define what students are to do with the mathematics content, are often too broad and not measurable. If a teacher is expected to teach in a standards-based environment, then the teacher has to be able to make judgments about the extent to which students have met the standard. Verbs that define observable, measurable activities make this possible.
- Washington's standards rely on examples. Many, perhaps most, of the GLEs require the user to read the examples to decipher what the standard means and requires.
- GLEs for grades 11 and 12 appear to be a combination of the Washington College Readiness Standards and NAEP. Grades 9-12 are missing most of algebra II, geometry and trigonometry. Whether or not this is the genesis of the grade 11 and 12 standards, they do not define a threshold for mathematical capability.

Recommendations

Recommendation 1: *Set higher expectations for Washington's students by fortifying content and increasing rigor.*

Washington's standards were first published in 1997. They reflected the then widely embraced standards of the National Council of Mathematics Teachers. Washington was in the forefront as it built an aligned and comprehensive set of standards, assessment, and professional development system to support student achievement.

Much more is now known about standards that support student learning than a decade ago. Washington is wise to revisit its standards and learn from the work of others.

Well-defined content is the core of a standards document. The content in Washington's standards needs to be strengthened in three ways.

Introduce students to core content at earlier grade levels, expand the content to include more advanced topics, and better develop core concepts, especially at the secondary level.

First, the GLEs need to be expanded to incorporate missing content. For instance, the concept of "odd and even" is missing in Washington but present in both Indiana and Massachusetts. This important early concept helps prepare students for division, prime numbers, factorization, and prime roots. Although it is likely that this concept is widely taught by Washington teachers, it should be clearly stated in the GLEs.

Second, much of the core content in the standards needs to be moved to earlier grade levels. Fractions, for instance, are introduced in Washington at grade 4. Singapore and California introduce the concept and use of fractions at grade 2.

Another example is that Washington expects students to add and subtract numbers to 18 in grade 2. Singapore introduces adding and subtracting in primary 1 and expects students in primary 2 to add and subtract numbers to 1,000. Since students in

Singapore primary 1 are the same age as students in grade 1, this is a marked difference.

Quadratic equations, discussed later in this paper, provide one last example. They are not mentioned in Washington until grades 11 and 12. Some states include quadratic equations at grade 8, and most expect mastery at grade 9.

We are not suggesting that standards from other places should be adopted wholesale but that the Standards Revision Team considers expectations from other places during the revision process.

Third, some of the topics included in Washington's standards need to be better developed. The quadratic equation, a fundamental tool in solving algebraic equations, exemplifies an underdeveloped topic. As mentioned, Washington's standards first reference the quadratic equation in the grade 11 and 12 GLEs. Even at this late stage of a student's education, the GLEs fail to treat it with necessary depth or breadth. Compare how the quadratic equation is treated in Washington and in Indiana.

Washington

- Apply procedures to solve linear inequalities, quadratic, absolute value, radical, and exponential equations.
- Define symbolic notation of functions (linear, quadratic, cubic, simple exponential, and simple rational functions).

Indiana

- Graph quadratic, cubic, and radical equations.
  Example: Draw the graph of $y = x^2 - 3x + 2$. Using a graphing calculator or a spreadsheet (generate a data set), display the graph to check your work.
- Solve quadratic equations by factoring.
  Example: Solve the equation $x^2 - 3x + 2 = 0$ by factoring.
- Solve quadratic equations in which a perfect square equals a constant.
  Example: Solve the equation $(x - 7)^2 = 64$.
- Complete the square to solve quadratic equations.
  Example: Solve the equation $x^2 - 7x + 9 = 0$ by completing the square.

- Derive the quadratic formula by completing the square. Example: Prove that the equation $ax^2 + bx + c = 0$ has solutions $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- Solve quadratic equations using the quadratic formula. Example: Solve the equation $x^2 - 7x + 9 = 0$.
- Use quadratic equations to solve word problems. Example: A ball falls so that its distance above the ground can be modeled by the equation $s = 100 - 16t^2$, where s is the distance above the ground in feet, and t is the time in seconds. According to this model, at what time does the ball hit the ground?
- Use graphing technology to find approximate solutions of quadratic and cubic equations. Example: Use a graphing calculator to solve $3x^2 - 5x - 1 = 0$ to the nearest tenth.

The examples, "Evidence of Learning," listed under the GLEs contain some of the important mathematics content included in Indiana. The Standards Revision Team can look to the content in the examples when they revise the standards in the algebra strand and throughout the document.

During our review, we found specific areas in need of strengthening that we identify for OSPI to review as it revises Washington's standards.

Kindergarten to grade 8:

- Anchor basic arithmetic operations in the place value system — the foundation of our numbering system.
- Strengthen the development of addition and subtraction concepts by linking them to place value, grouping, and regrouping.
- Memorize the single-digit addition and the corresponding subtraction facts and memorize the single-digit multiplication facts and their corresponding division facts.
- Place a higher priority on fractions, and present them in a mathematically structured, coherent fashion beginning in early elementary grades.
- Fortify the grade 3 curriculum by developing students' understanding of the concepts of numbers written out using place value by applying the commutative and the distributive property to see how the standard algorithm for multiplication comes about naturally.

- Develop, beginning in grade 4, the standard long division algorithm. Long division builds estimation skills and sets up students for a basic concept in calculus: successive approximation. Long division does not always give you a final answer, but each step gives you a better approximation of the answer.
- Stress multi-step word problems for all K–8 mathematics.

All graduating students:

- Refashion the high school algebra standards to include the concepts contained in the GLE examples for Algebra I. The examples under the GLEs contain concepts and skills worthy of GLE status. These should not be suggestions, but requirements.
- Develop high school geometry in a coherent fashion with a good introduction to proofs, including such foundational concepts as axioms, postulates, and beginning graph theory.

Grades 11 and 12:

The following content is for students who take the equivalent of algebra II and trigonometry:

- Reinforce the essential standard of being able to manipulate rational functions by spelling out student requirements for the application of procedures to simplify and evaluate polynomial, rational, absolute value, and radical expressions, as well as using the four arithmetic operations with these expressions.
- Expand the standards for quadratic equations to include a deep analysis of line of symmetry, max/min for quadratic functions, the quadratic formula, and completing the square to determine the attributes of a circle, ellipse, parabola, and hyperbola.
- Add complex numbers, polar coordinates, and induction.
- Incorporate function composition and inverse functions, matrices, conic sections, advanced statistics, logarithms, and series. These topics are contained in at least two of the following documents important to Washington and its students: Washington College Readiness Standards, American Diploma Project Benchmarks, and the National Assessment for Education Progress.
- Include a study of trigonometry supported by basic trigonometric functions used in computations, such as sine, cosine, and tangent.

The rigor in Washington's standards is compromised by two factors.

First, as just discussed, mastery of content is sometimes expected late in a student's career.

Second, the rigor inadvertently is undermined by the overuse of the verb "understand." "Understand" does not necessarily require students to use mathematics content in sophisticated ways. Yes, students must understand. They also need to be able to demonstrate their understanding in explicit ways, including solving multi-step word problems.

To illustrate, in grade 4, EALR 1, "understand" is used 10 times, "use" three times, "apply" twice, and "read" and "recognize" once each. Contrast this with the same grade level in California. California expects students to write, order, round, explain, solve, estimate, manipulate, and compute.

Recommendations to fortify content and increase rigor:

- ***Include more content and better develop some of the content already included.***
- ***Move appropriate content to lower grades.***
- ***Reduce the use of the verb "understand," and replace it with observable verbs that require students to do demanding work.***

Recommendation 2: *Make clear the importance of all aspects of mathematics: mathematics content including the standard algorithms; conceptual understanding of the content; and the application of mathematical processes within the content.*

Mathematics can be thought of as a three-legged stool: mathematical content, including standard algorithms (step-by-step procedures that work in every situation, are transparent to the user, and generalize to higher mathematics,) conceptual understanding, and mathematical processes (communicating, reasoning, problem solving, and connecting mathematics content). Missing any leg cripples the user. Applying content in novel situations without conceptual understanding is close to impossible. Applying conceptual understanding in high-level mathematics without algorithms is impossible.

One of the strengths of the current GLEs lies in the maturity of the process strands. They are thoughtfully constructed and well developed.

Washington also makes clear the importance of conceptual understanding with numerous standards like "Understand the concept of area." GLEs like this show that Washington values knowledge about the principles of mathematics and why mathematics works as it does.

The value Washington places on mathematics content and standard algorithms is not as apparent. Certainly the case can be made that the standards are built on content and that algorithms are implicit. We would say that this needs to be more obvious. There should be a clear mandate to teach transparent, efficient algorithms that work beyond simple numerical situations and are connected to well-defined content.

Mathematical content and fluency with standard algorithms, conceptual understanding, and mathematical processes are all necessary. This should be crystal clear in Washington's standards.

The rich development of mathematical processes and conceptual understanding need not distract from the content; on the contrary, the focus of conceptual understanding should be on the content.

Many GLEs are similar to this one from grade 4: "Apply strategies, and use tools appropriate to tasks involving multiplication and division of whole numbers."

"Strategies" gives the impression that multiplication and division problems are *ad hoc* problems. They are not. There are standard procedures, algorithms that work every time. Let students look for many ways to solve and model simple multiplication problems; this builds understanding. Just make sure that in the end they have mastered the standard algorithms in the standard notation. These algorithms always work, are efficient, give the students and teachers a common language, and are designed to generalize to new situations in higher mathematics. These traits are particularly relevant if students go to college.

From the same example, the phrase “use tools appropriate,” begs the questions of what kind of tools and under what conditions they should be used.

Algorithms are tools, yet nowhere are students asked to memorize their addition and multiplication facts (also a tool) or use standard algorithms with fluency and efficiency. These algorithms should be taught and listed as explicit standards.

There is a belief that calculators are used in elementary classrooms in lieu of memorizing mathematics facts. There are plenty of anecdotes but no research to support this. It is known that American teachers use calculators in elementary schools more often than their counterparts in high achieving countries[i], but that does not necessarily equate to teaching students to use calculators rather than memorizing mathematics facts and standard algorithms.

Calculators have value in the elementary classroom — but not to replace computation. Students need to add, subtract, multiply, and divide whole numbers, fractions, and decimals without a calculator.

Calculators have value in the elementary classroom — but not to replace computation. Students need to add, subtract, multiply, and divide without a calculator.

At the secondary level, again, the use of technology should not circumvent student fluency with hand calculations. Graphing calculators and spreadsheets are useful tools that help students make connections between different representations. Many concepts — range, roots, and optimum values — come alive with technology. A variety of available software helps students understand systems of linear equations, quadratic functions, conics, and more. Still, the use of technology should not replace a student’s ability to solve problems manually.

The use and misuse of technology mandates plain direction in the standards.

Recommendations to clarify the importance of all aspects of mathematics: mathematical content and algorithms, the conceptual understanding of both, and the application of processes within the content:

- ***Require students to memorize mathematics facts and use standard algorithms fluently.***
- ***Continue to emphasis the importance of understanding mathematics.***
- ***Retain the richness of the process strands, in part, by thoughtfully embedding them into important content.***

Recommendation 3: *Identify those topics that should be taught for extended periods at each grade level, and better show how topics develop over grade levels.*

The existing document, organized by components, shows well how existing topics in GLEs develop over and connect within grade levels. But the EALR format fosters "form over structure" and encourages creating a GLE for every "cell," meaning every topic is equally emphasized at every grade level. Additionally, the content for a single topic is spread over multiple grade levels, and every grade or course includes numerous topics. This leads to a "mile wide, inch deep" approach to teaching.[ii] While a few cells are intentionally left blank, the vast majority is filled in, even when it convolutes the content.

When the standards originally were adopted 10 years ago, building concepts over grades and revisiting topics repeatedly were believed to be the best ways to ensure student understanding, remembering, and expertise. This is no longer considered best practice for teaching and learning.

Currently, there is no way, beyond a teacher's good sense, to know whether he or she should pay more attention in grade 5 to mixed numbers, proper and improper fractions, and decimals; angle measurement; or mean, median, and mode.

The revised standards should identify topics that will be taught over extended periods of time during a single school year in order to give students the opportunity to develop deep understanding and to truly master the content. These extended-time topics or grade-level "themes" should be clearly defined. Teachers should not have to guess which topics are most critical.

Currently, there is no way, beyond a teacher's good sense, to know which topics deserve extended teaching time.

Some non-priority topics should be consolidated and moved to single (or successive) grade levels. Rather than spreading statistics, for example, across all 12 grades, it should be taught for an extended period of time during two or three selected grade levels.

This approach allows topics to have depth and to be developed fully.

Grade-to-grade coherence — the extent to which a topic's complexity grows sequentially over grade levels — relates to prioritizing topics. Consolidating topics within and over fewer grades makes the subsequent development more apparent.

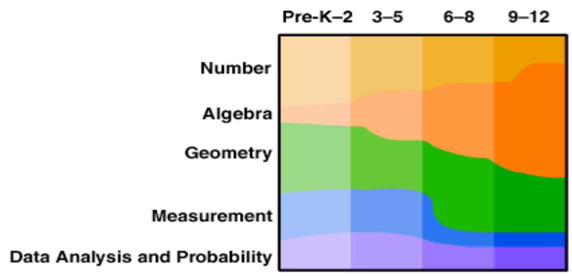
We are not suggesting that all topics except the priority topics be excluded. Telling time, for example, is and should be part of a mathematics curriculum. It just should not take up much time in that curriculum. It can be reinforced by bringing it into the priority topics by asking, "How many minutes pass between 9:18 a.m. and 2:15 p.m.?"

The present structure also means that single topics get distributed to various nooks and crannies of the document. For example, "fractions" appears to be well developed, but it is so scattered throughout the document that it takes more time than it should to determine this. The structure of mathematics is part of the content of mathematics. To present content properly, the structure of mathematics needs to be clear.

At the risk of being repetitive, the foundation of mathematics is number sense and computation, and these topics should be emphasized in elementary grades. Place value makes possible arithmetic as we know it. (Think of trying to compute with Roman numerals.) Adding, subtracting, multiplying, and dividing whole numbers, fractions, and decimals without the aid of a calculator and with fluency should be the core of K–5 mathematics. These should be followed in subsequent grades with ratios, rates, proportions, and percentages. Washington should insist on the ability of all students to use all of these skills to solve multi-step word and story problems.

There are several resources that offer ways to expand, reduce, and consolidate mathematics topics.

Principles and Standards, published in 2000 by the National Council of Teachers of Mathematics (NCTM), includes this chart to clarify its position that not all topics are equally important at all grade levels.



Curricular Focal Points (CFP), released by NCTM in September 2006, identifies major topics at each grade level K–8. This document has been well received by people with differing perspectives and is probably the best resource for this issue. *CFP* limited itself to three core topics per grade level and identified how these connected to other topics in mathematics.

Singapore varies the strands that are taught at different grade levels. Some topics are consistent across grade levels, whereas other topics fade away to be replaced with new topics.

The chart below shows selected grade levels from Singapore.

Content topics vary across grade levels in Singapore

Grade 2	Grade 6	Grades 9 and 10
Whole numbers Money, Measurement and Mensuration (the measurement of geometric figures) Statistics Geometry Fractions	Money, Measurement and Mensuration Statistics Geometry	Numbers and Algebra Geometry and Measurement Statistics Probability

Indiana uses number sense and operations; patterns, relations, and algebra; geometry; measurement; and data analysis, statistics, and probability as the organizational structure for grades K–12. At the secondary level, it also offers the standards by course level: Algebra I, Geometry, Algebra II, and Pre-calculus. Other states use variations on this organizational system.

Recommendations to identify those topics that should be taught for extended periods at each grade level and to better show how topics develop over grade levels.

- ***Major themes that require extended teaching time should be identified for each grade level. The topics in need of front-and-center attention for both elementary and secondary are identified in the previous section.***
- ***Grade-to-grade coherence of mathematical topics should be apparent with ever more sophisticated content appearing over grade levels.***
- ***Additional topics that are more minor in nature should be explicitly included while making it clear they require less classroom time.***
- ***Some topics should be consolidated and moved to single grade levels where they will take their turn as areas of emphasis.***

Recommendation 4: *Increase the clarity, specificity, and measurability of the Grade Level Expectations***.**

Many of the current GLEs lack clarity. This may have been an intentional attempt to allow for rich interpretation by teachers, but the side effect is that some GLEs are open to so many interpretations that its impossible to be certain of their meaning. This compromises the intent of standards, which is to define clearly what should be taught, when.

Another way that the GLEs are too broad and vague is shown by this grade 5 example: "Apply strategies, and use tools appropriate to tasks involving addition and subtraction of non-negative decimals or like-denominator fractions." Terms like "strategies," "tools," and "as appropriate" are undefined and muddy understanding.

In too many GLEs, the verbs are broad and unobservable, and the content is ill defined. Simply put, there is not enough detail — specificity — for the GLEs to be clear. Certainly this is not always the case, but it is persistent and needs to be addressed. Standards should be precisely written in unambiguous language.

The following chart shows how Washington often relies on vague verbs and broadly defined content by comparing the GLEs to standards from other documents. Verbs are underlined to make the differences apparent.

WA	CA	MA	IN	Curricular Focal Points
1.1.1 Understand the concept of number to at least 31.	**N 1.2** Count, recognize, represent, name, and order a number of objects (up to 30).	**K.N.1** Count by ones to at least 20. **K.N.2** Match quantities up to at least 10 with numerals and words. **K.N.8** Estimate the number of objects in a group and verify results.	**K.1.3** Know that larger numbers describe sets with more objects in them than sets described by smaller numbers. **K.1.6** Count, recognize, represent, name, and order a number of objects (up to 10). **K.1.8** Use correctly the words one/many, none/some/all, more/less, and most/least.	**N 1** Children use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set, creating a set with a given number of objects, comparing and ordering sets or numerals by using both cardinal and ordinal meanings, and modeling simple joining and separating situations with objects. **N 2** They choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the number in a small set, counting and producing sets of given sizes, counting the number in combined sets, and counting backward.

It is not easy to create a set of standards with enough — but not too much — detail. Striking the right balance between broad statements that are open to multiple interpretations and overly detailed standards that reduce mathematical topics into a set of tiny, discrete skills is difficult.

The old adage that "things should be as simple as possible, but no simpler" applies here.

Often, the details embedded in other standards documents are included in the examples under the GLEs. The following chart shows that Indiana's standards are written with about the same level of detail as are Washington's examples. While the examples definitely add meaning to the GLEs, what is needed are GLEs that stand alone.

Indiana further enhances the intentions of its standards by embedding sample problems.

WA, grade 4	Related Indiana standards from grades 3 and 4
1.1.5 Understand the meaning of addition and subtraction of like-denominator fractions. EXAMPLES EX Represent addition and subtraction of fractions with like denominators using numbers, pictures, and models, including everyday objects, fraction circles, number lines, and geoboards. EX Use joining, separating, part-part-whole, and comparison situations to add and subtract like-denominator fractions. EX Translate a given picture or illustration into an equivalent symbolic representation of addition and subtraction of like-denominator fractions. EX Select and/or use an appropriate operation to show understanding of addition and subtraction of like-denominator fractions.	▪ Show equivalent fractions using equal parts. *Example: Draw pictures to show that 3/5, 6/10, and 9/15 are equivalent fractions.* ▪ Identify and use correct names for numerators and denominators. *Example: In the fraction 3/5, name the numerator and denominator.* ▪ Given a pair of fractions, decide which is larger or smaller by using objects. *Example: Is 3/4 of a medium pizza larger or smaller than 1/2 of a medium pizza? Explain your answer.* ▪ Rename and rewrite whole numbers as fractions. *Example: 3 = 6/2 = 9/3 = ?/4 = ?/5.* ▪ Name and write mixed numbers, using objects or pictures. *Example: You have 5 whole straws and half a straw. Write the number that represents these objects.* ▪ Name and write mixed numbers as improper fractions, using objects or pictures. *Example: Use a picture of 3 rectangles, each divided into 5 equal pieces, to write 2 3/5 as an improper fraction.*

The good news is that addressing specificity and clarity by adding detail to the content and replacing "understand" with observable, measurable, action verbs creates a set of standards that teachers can use to gauge student progress and testing companies can use to design appropriate test items.

Specific recommendations for clarity, measurability, and specificity are:

- ***Define the parameters of the content with details that provide clear guidance.***
- ***Replace the verb "understand" with observable, action verbs that describe what students should be able to do with the content.***
- ***Define when and under what circumstances technology, especially calculators, should be used. It must be clear that calculators should not be used in lieu of computational mastery.***
- ***Embed sample problems within the GLE as needed for clarity.***

Recommendation 5: *Write Essential Academic Learning Requirements that restructure the standards to clarify grade-level priorities and that reflect both the conceptual and procedural sides of mathematics.*

Washington's current structure, with five EALRs that are consistent across all grades, needs to be modified. Crafting new EALRs that restructure the standards allows Washington to reflect both the conceptual and procedural aspects of mathematics and to distinguish topics that deserve extended teaching time. The mathematics identified in Recommendation 1 offer a core for this work.

The goal is to identify a limited set of topics that can be taught and learned well within a school year.

The topics or strands defined by the EALRs need not be consistent across all grade levels. It is more important that they assign critical content to specific grade levels and that they ensure that related content is all together to preserve the structure of mathematics. The document should leave no room for confusion about what is important.

To help accomplish this, we suggest unpacking EALR 1, which holds the mathematics content, to create separate EALRs that define topical strands. The new EALRs can vary across grade bands to define and highlight important content at specific grades.

We also suggest collapsing the process strands into fewer EALRs. We like the idea of reducing the number of EALRs from four to two: 1) Reasoning and problem solving and 2) Communication.

Here is the reasoning for this suggestion. There is so much overlap that is impossible to distinguish between the first two, "reasoning" and "problem solving," so they should be combined. "Communication" should continue to stand alone. The idea of mathematical "connections" is well intended but very difficult to characterize and generalize across all content strands. The best way to handle this EALR is to thoughtfully embed it within the content EALRs when appropriate.

Content, processes, and understanding intertwine; defining what they are and the relationships among them is the work of writing standards.

Conceptual understanding should also be evident in the restructured document. Content, processes, and understanding intertwine; defining what they are and the relationships among them is the work of writing standards. It is both necessary and difficult. For example, a well-written sequence of standards about multiplication should include building understanding of what multiplication is, single-digit multiplication facts, the standard algorithm, and how the standard algorithm comes about from place value, distributivity, and commutativity.

Embedding conceptual understanding and mathematical processes into the content strands shows how and when they intersect. Explicitly connecting process to content reinforces the importance of each.

The following table illustrates how GLEs could be strengthened by explicitly weaving suitable processes into the most appropriate content.

WA, grade 4:	**Indiana, grade 4:**
Apply concepts and procedures from any two of the content strands, including number sense, measurement, geometric sense, statistics, and/or algebraic sense, in a given problem or situation.	Recognize and apply the relationships between addition and multiplication, between subtraction and division, and the inverse relationship between multiplication and division to solve problems.
Rather than asking students to connect everything to everything, Indiana makes explicit those times when it is most important for students understand how specific mathematics relate to each other. The relationships among addition, subtraction, multiplication, and division are one instance.	
Understand how to organize numerical, measurement, geometric, and/or statistical information to communicate for a given purpose.	Summarize and display the results of probability experiments in a clear and organized way. *Example: Roll a number cube 36 times and keep a tally of the number of times that 1, 2, 3, 4, 5, and 6 appear. Draw a bar graph to show your results.*
Rather than a general expectation that students communicate in all content areas (which should continue to be included in the EALR on communication), Indiana stresses the importance of communication as it relates to specific standards, such as this one about probability.	

Recommendations for writing EALRs that restructure Washington's standards to clarify grade-level priorities and reflect both the conceptual and procedural sides of mathematics:

- ***Create an expanded set of content-based EALRs.***
- ***Collapse EALRs 2–5 into fewer process strands.***
- ***Allow the number of content strands to vary across grade levels.***
- ***Embed conceptual understanding and mathematical processes into the content strands.***

Recommendation 6: *Create a standards document that is easily used by most people.*

The current, multigrade format is helpful to curriculum specialists who work across grade levels, groups of teachers when they meet to talk about how content builds across grade levels and the implications of that on teaching, and individual teachers when they want to see what is taught before and after their grade level.

It is less helpful to other audiences and for other purposes.

An alternate format should offer GLEs by single grade levels for the teachers who want a concise outline of their responsibilities and for parents who may be interested only in the standards for their student's grade level.

To enhance relevancy for high school teachers, the standards should be offered in two ways.

Secondary GLEs should be organized by traditional courses in one format. Currently, at least to the outsider, it is impossible to tell what standards should be taught in such courses as Algebra and Geometry.

Offer GLEs by grade level, by course, and in the current format.

GLEs also should also continue to be organized by grade level. The current, grade-by-grade format may be a better fit for those who teach Integrated Math I, II, III, and IV, since each of these courses matches to a single grade level. We are not certain this is the case. We do not know that the GLEs in grades 9 and 10, for example, match the topics taught in Integrated I and II.

Students should be guaranteed that no matter which way content is taught, they will see the standards necessary for graduation.

Massachusetts offers a helpful example. Its secondary standards are offered both by grade level and by course. And it cross references the standards by using their identification number so that it is easy to see where "solve linear equations", for example, is taught in the Integrated series and in Algebra. This makes it is clear that a student taking Integrated Math series of I, II, and III will learn the same materials as a student taking Algebra I, Geometry, and Algebra II.

Washington's standards document can be made more useful to students who want to go beyond the expectations required for graduation. GLEs should be expanded to include the content of algebra II, trigonometry, and perhaps pre-calculus. These GLEs, typically taught in grades 11 and 12, prepare a student for college-level mathematics.

There are a variety of classes offered at the college level. Students may start their study of mathematics in college with a liberal arts mathematics classes or with calculus. The standards should identify the skills and knowledge necessary for success.

Minor changes — more technical than substantive — would make the document more transparent and useable.

Enrollment should not be restricted to those students planning to go to college; how many high school students really know what career they will pursue?

This need not encourage tracking; it is not about limiting the opportunity of able and interested students to take high-level mathematics courses. All students should be encouraged to take as much mathematics as possible.

The standards document should differentiate between the GLEs required for all students and content that exceeds those expectations with some type of demarcation. This suggestion clarifies the implementation of Recommendation 1.

Primarily, offer the GLEs in the following ways:

- By single grade levels K–12 to serve elementary teachers, secondary teachers of Integrated Math, and parents interested in what their children are learning.
- By secondary courses so that it is clear which standards should be taught and learned in specific classes.
- As is, showing multiple grades, to support vertical articulation and mathematics instruction.
- With GLEs that define the content necessary for post-secondary education.

Also consider offering other versions, which can be developed and offered over time:

- By mathematics strands so that the development of algebra, for example, can be traced over multiple grade levels.
- In additional levels of complexity, beyond the current version, which is geared to educators. A simply written standards documents could be offered to parents and students to answer such questions as, "What will my second-grade student learn this year?" And a more specific level could be created for audiences like the testing contractor. This version would include such things as assessment limits on types and sizes of numbers.
- In multiple languages, beginning with Spanish.
- With different illustrations and example problems for various audiences.

All versions and all levels should be available to everyone. This is about creating options and making those options available through a well-designed menu of choices to anyone who is interested.

Make minor changes in format to increase usefulness: Additional, relatively minor changes — more technical than substantive — would make the document more transparent and more useable.

- Online technology has come a long way since Washington's standards were written. Take advantage of this progress to offer a rich, multilayered document. Foremost, GLEs should be linked to corresponding textbooks and curriculum. Hyperlinks to definitions, example problems, released test items, and explanations of content also make good use of the technology.
- Renumber the GLEs so they are more intuitive. The Massachusetts system tells the user at a glance how the standard they are looking at fits into the schema. For instance, 4.N.3 is a grade 4 standard from the Number Sense strand, and it is the third standard listed.

- Identify, within the GLEs, words included in the glossary. This makes the standards easier to use. Providing definitions within the GLE text occasionally makes sense, but generally, noting terms that can be found in the glossary — perhaps with an asterisk — serves the purpose without adding unnecessary bulk.
- Consider aligning — precisely, standard by standard — with high school assessments such as SAT/PSAT/ACT/AP/End of Course assessments and college placement tests. This information does not have to clutter the main document but would be one of the supplemental versions available or available as online hyperlinks.

As previously discussed, Washington's standards are organized into five levels.

Each EALR has multiple components, each component has subcomponents, each subcomponent has multiple GLEs (which are further organized by grade level), and most GLEs have multiple examples.

1. Essential Academic Learning Requirement
2. Component
3. Subcomponents
4. Grade-Level Equivalent
5. Examples

One way to think of this hierarchy is that each EALR stands for what most states call content "strands." The components are like "substrands," the GLEs are analogous to what most states call "standards," and the examples serve to amplify the standard. Another way to think about the EALR hierarchy is to visualize each EALR as an overarching idea with the components as strands and the GLEs as standards, etc. Either way, it is confusing to users.

Most standards documents have three levels, although the American Diploma Project has only two. Appendix D shows Massachusetts' numbering system in which 4.N.1 equates to grade 4, Number Sense strand, first standard. Some sort of simple hierarchy with an intuitive structure is the backbone of a document that is easily used by as many people as possible.

Recommendations to increase accessibility:

- ***Offer the standards in a variety of formats. At the minimum, make them available by grade level, subject course name, and grade bands.***
- ***Use online technology with embedded hyperlinks that allow users to access additional information. Especially important is linking GLEs to the relevant curriculum. Showing the alignment between the GLEs and pertinent assessments has high value. Other information — such as additional sample problems, released test items, and explanations of content —also is useful.***
- ***Include and mark grades 11 and 12 GLEs that prepare a student to be successful in post-secondary education.***
- ***Renumber the GLEs in a way that is more intuitive.***
- ***Embed indicators that alert the reader when a word is included in the glossary.***
- ***Reduce the number of "levels" in the current document.***
- ***Consider offering other formats, such as by content strand or different levels of complexity.***

Recommendation 7: *Create small, expert Standards Revision Teams, and systematically collect feedback on the revised standards.*

The OSPI has the responsibility of revising Washington's mathematics standards within a few months. This is a short time frame for such work.

The best way to go about this is to create small teams for each grade band. The teams need to include the people most necessary to the success of the work: a mathematician, a mathematics educator, a teacher from the relevant grade band, and a curriculum specialist. One person with extensive standards experience in multiple states should facilitate, organize, and coordinate the work to be sure there is consistency across grade bands.

Other perspectives — from business community members, Transition Math Project members, parents, mathematics educators, college educators, industry leaders, child development experts, and mathematics researchers — are valuable and also should be heard. But if OSPI is to keep the ambitious schedule set by the legislature, the writing teams must be small.

In lieu of inclusive writing teams, OSPI should convene formal focus groups to listen to other stakeholders. Targeted focus groups — one for mathematicians, mathematics educators, and mathematics researchers; one for the business community that includes representatives from mathematics-intensive fields; one for teachers; one for parents; and one for students — should inform the standards revision process.

While all groups are critical, having mathematicians review the revised standards is vital. Washington needs to be sure the mathematics in the document is accurate.

Additionally, the draft standards should be sent to groups, such as the Transition Math Project, that have special interests in the standards and valuable input to offer.

All of this would be in addition to normal feedback loops, such as posting the document on the Internet.

Recommendations to create small, expert Standards Revision Teams for each grade band and to systematically gather feedback on the revised standards:

- ***Create separate writing teams for each grade band that includes a mathematician, a mathematics educator, a curriculum specialist, and a grade-appropriate teacher. Coordinate the efforts of the three groups with one national expert.***
- ***Hold at least four focus groups for critical stakeholders.***
- ***Send the draft document to interested groups.***
- ***Offer normal feedback loops, such as online posting.***

Conclusion

The State Board of Education is wise to commission an independent review of its mathematics standards. Although, as we said in the beginning, standards are only one piece of a complex, interconnected system; they are the corner stone. As such, they (as well as the standards for each of the content areas) warrant an independent review on a set schedule of between five and seven years.

We offer a clear and specific set of recommendations for revising Washington's mathematics standards. We believe that these recommendations will produce a strong set of standards to serve as the base for the work outlined in the Joint Mathematics Action Plan. As we said in the beginning, Washington is moving in the right direction. The revised standards are the first step in accelerating that progress.

Appendix A

DETAILS OF DOCUMENTS USED FOR COMPARING GLEs BY GRADE LEVEL

	K	2	4	6	8	10	12
CA	**K**	Grade 2	Grade 4	Grade 6	Algebra I	Algebra I and Geometry	Algebra II
MA	Grade band pre-K and K	Grade band 1–2	Grade band 3–4	Grade band 5–6	Grade band 7–8	Grade band 9–10	Grade band 11–12
IN	K	2	4	6	8	Integrated 1 and 2	Integrated 3
SG	Doesn't exist at this grade	Primary 2	Primary 4	Primary 6 EM1/EM2 iii	"O" level Secondary 1	"O" level Secondary 2	"O" level Secondary 3/4
FI	Doesn't exist at this grade	Grade 2, "Core content" and "Measures of good perform-ance"	Content is organized in grade band 2–5	Content is organized in grade band 6–9	Compared to grade 8 assess — when working from GLE examine grade band 6–9 and do not score for rigor	Not available at time of review	Not available at time of review
CFP	Break the paragraphs into sentences and skip "connec-tions"	Break the paragraphs into sentences and skip "connec-tions"	Break the paragraphs into sentences and skip "connec-tions"	Break the paragraphs into sentences and skip "connec-tions"	Break the paragraphs into sentences and skip "connec-tions"	Doesn't exist at this grade	Doesn't exist at this grade

CA	California
MA	Massachusetts
IN	Indiana
SG	Singapore
FI	Finland
CFP	Curricular Focal Points

	K	2	4	6	8	10	12
NAEP	Doesn't exist at this grade	Doesn't exist at this grade	Grade 4	Doesn't exist at this grade	Grade 8	Doesn't exist at this grade	Grade 12
ADP	Doesn't exist at this grade	Doesn't exist at this grade	Doesn't exist at this grade	Doesn't exist at this grade	Doesn't exist at this grade	Doesn't exist at this grade	Benchmark here
CRS	Doesn't exist at this grade	Doesn't exist at this grade	Doesn't exist at this grade	Doesn't exist at this grade	Doesn't exist at this grade	Doesn't exist at this grade	Benchmark here

NAEP	National Assessment for Educational Progress
ADP	American Diploma Project
CRS	College Readiness Standards (WA)

Appendix B

RUBRIC SUMMARIES

Key ideas, by score point, for content, rigor, specificity, and clarity

Score	Content	Rigor	Specificity	Clarity
4	Regardless of grade level *or* cognitive demand; Content matches content. Note: Wording may not match exactly.	The content and grade level match *and* the GLE Bloom's Taxonomy level is at least as demanding: 1. Remembering 2. Understanding 3. Applying 4. Analyzing 5. Evaluating 6. Creating.	The level of detail in the Benchmark matches the detail in the GLE. Matching grain size.	Reader finds total clarity. Content and parameters of content well-defined. Standard well-written and clear. No ambiguity.
3	Regardless of grade level or cognitive demand; Content is connected but not an exact match.	The content matches and the grade levels match … *but* the GLE taxonomy level is lower than the Benchmark.	Missing detail matches logical assumptions Similar, but not matching, grain size.	Understandable; stands alone; good vocabulary. May have small disputes in meaning.
2	Regardless of grade level or cognitive demand; Content is present but not as sophisticated.	The GLE content is at a lower grade … … EVEN WHEN the GLE taxonomy level is equal to or higher than the Benchmark.	The level of detail of the GLE differs in regard to content or performance demands of the Benchmark. Different grain size.	Multiple possible meanings; vague wording. Examples clarify somewhat.
1	Content *not* present in GLE.	The GLE content is at a lower grade … and the taxonomy level is lower than the Benchmark.	The topic or sub-topic detail in the GLE does not approach the scope or topic detail in the Benchmark. Dissimilar scope.	Unsure of meaning even after Examples and discussion. Incorrect use of mathematical vocabulary automatically scores 1.
NA	Missing Content.			Missing Content.

Key ideas, by score point, for depth, grade-to-grade coherence, measurability, accessibility, and balance

Score	Depth	Grade-to-Grade Coherence	Measurability	Accessibility	Balance
4	All major topics are included and are fully developed.	Content systematically increases in complexity. Prerequisites are in place. Consistent cognitive demand across strands with evenly distributed content across grade levels.	Observable content, skills, behaviors. Clear expectations for mastery.	Useable by as many people as possible. Format is self-explanatory. Content available in a variety of ways.	Addresses all three aspects: mathematics concepts, algorithms, and why the algorithms work.
3	All major topics are included *but* a few topics are underdeveloped.	Content increases in complexity, but there are missing prerequisites; *or* Inconsistent expectations around cognitive demand; *or* Problems with content distribution.	Clearly defined content. Murky expectations of performance.	Generally understandable; Only one format.	Addresses both conceptual understanding and execution.
2	Most major topics are included *and* a few topics are underdeveloped.	Minor breakdowns in content sequence. Two or more of the problems in 3.	Content defined. Verbs are difficult to assess.	Understandable to teachers but difficult for others.	Both concepts and algorithms are present, but one dominates.
1	Numerous topics are missing.	Serious inconsistencies or interruptions in the content sequence.	Content vaguely defined. Unobservable verbs.	Difficult to understand by everyone.	One aspect is excluded.

Appendix C

MEMBERS OF THE STANDARDS REVIEW TEAM

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Sandy Sanford, Ed.D., has been teaching in California public schools since 1989. He has taught at all levels and ended his public school career as district administrator of assessment, research, and evaluation. In 2000, Dr. Sanford created a laboratory to develop better methods for gathering, analyzing, reporting, and using assessment data to guide the instructional process in a standards-based educational environment. Sandy has an M.S. in systems management from the University of Southern California and a M.Ed. and an Ed.D. from Azusa Pacific University.

Eric J. Rawdon, Ph.D., is an assistant professor at the University of St. Thomas, where he has taught Calculus I and Multi-Variable Calculus. He received his B.A., cum laude, from St. Olaf College and his Ph.D. from the University of Iowa. His specialty is topology and computational mathematics (physical knot theory).

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Appendix D

STATE STANDARDS FORMAT SAMPLES

Massachusetts Standards: Excerpt from Pre-K–K and 1–2

Number Sense and Operations

✚ Understand numbers, ways of representing numbers, relationships among numbers, and number systems.

● Understand meanings of operations and how they relate to one another.

▲ Compute fluently and make reasonable estimates.

Grades Pre-K–K

Learning Standards	**Selected Problems or Classroom Activities**
Students engage in problem solving, communicating, reasoning, connecting, and representing as they: K.N.1 Count by ones to at least 20. ✚ K.N.2 Match quantities up to at least 10 with numerals and words.✚ K.N.3 Identify positions of objects in sequences (e.g., first, second) up to fifth. ✚ K.N.4 Compare sets of up to at least 10 concrete objects using appropriate language (e.g., none, more than, fewer than, same number of, one more than) and order numbers. ✚ K.N.5 Understand the concepts of whole and half. ✚ K.N.6 Identify U.S. coins by name. ✚ K.N.7 Use objects and drawings to model and solve related addition and subtraction problems to 10. ● K.N.8 Estimate the number of objects in a group, and verify results. ▲	*Refers to standards K.N.1–K.N.5* Listen to stories, and explore books that incorporate number concepts. *Refers to standards K.N.1, K.N.2, K.N.4, and K.N.8* Have each child estimate the number of seeds in a slice of watermelon by inspection. Remove and count the seeds and compare the estimate to the count. Children then draw and color pictures of slices of watermelon, paste the seeds on their drawings, record the number of seeds, and compare their watermelon slices to tell who has more seeds. *Refers to standards K.N.2 and K.N.3* Engage in games, songs, nursery rhymes, and dances that incorporate number sequences.

Exploratory Concepts and Skills

- ✓ Count by ones, beginning from any number in the counting sequence.
- ✓ Represent quantities using concrete objects, and investigate the portioning of sets. Identify equal parts of groups.
- ✓ Create problems that can be solved using addition and subtraction, multiplication and division, such as equal groupings of objects and sharing equally.

Grades 1–2

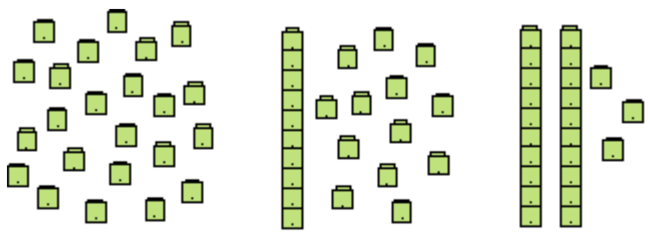
Learning Standards	**Selected Problems or Classroom Activities**
Students engage in problem solving, communicating, reasoning, connecting, and representing as they: 2.N.1 Name and write (in numerals) whole numbers to 1,000, identify the place values of the digits, and order the numbers. ✚ 2.N.2 Identify and distinguish among multiple uses of numbers, including cardinal (to tell how many) and ordinal (to tell which one in an ordered list), and numbers as labels and as measurements. ✚ 2.N.3 Identify and represent common fractions (1/2, 1/3, 1/4) as parts of wholes, parts of groups, and numbers on the number line. ✚ 2.N.4 Compare whole numbers using terms and symbols, e.g., less than, equal to, greater than (<, =, >). ✚ 2.N.5 Identify odd and even numbers, and determine whether a set of objects has an odd or even number of elements. ✚ 2.N.6 Identify the value of all U.S. coins, and \$1, \$5, \$10, and \$20 bills. Find the value of a collection of coins and dollar bills and different ways to represent an amount of money up to \$5. Use appropriate notation, e.g., 69¢, \$1.35. ✚ 2.N.7 Demonstrate an understanding of various meanings of addition and subtraction, e.g., addition as combination (plus, combined with, more); subtraction as comparison (how much less, how much more); equalizing (how many more are needed to make these equal); and separation (how much remaining). ● 2.N.8 Understand and use the inverse relationship between addition and subtraction (e.g., 8 + 6 = 14 is equivalent to 14 – 6 = 8 and is also equivalent to 14 – 8 = 6) to solve problems and check solutions. ● 2.N.9 Know addition facts (addends to ten) and related subtraction facts, and use them to solve problems. ▲ 2.N.10 Demonstrate the ability to add and subtract three-digit numbers accurately and efficiently. ▲ 2.N.11 Demonstrate in the classroom an understanding of and the ability to use the conventional algorithms for addition (two 3-digit numbers and three 2-digit numbers) and subtraction (two 3-digit numbers). ▲ 2.N.12 Estimate, calculate, and solve problems involving addition and subtraction of two-digit numbers. Describe differences between estimates and actual calculations. ▲	See next page for sample problems.

Exploratory Concepts and Skills

- ✓ Use concrete materials to investigate situations that lead to multiplication and division.
- ✓ Develop and use strategies for addition and subtraction of multi-digit whole numbers. Check by estimation.
- ✓ Investigate addition of common fractions, e.g., 1/2 + 1/2 = 1, 1/4 + 1/4 = 1/2.
- ✓ Understand situations that entail multiplication and division, such as equal groupings of objects and sharing equally.

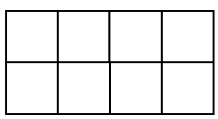
Refers to standard 2.N.1
Use 8, 6, and 4.
Write the smallest three-digit number: __
Write the greatest three-digit number: __
Write other numbers using the same digits: _______ _______ _______ _______

Refers to standard 2.N.1†



(a) 23 ones　　(b) 1 ten and 13 ones　　(c) 2 tens and 3 ones

Refers to standard 2.N.3



Count the small squares, and color 1/4 of them.

Refers to standard 2.N.6
P N N P N N . . .
P stands for penny, and N stands for nickel. If the pattern continues until there are 12 coins altogether, what is the total value of all 12 coins?

Indiana Standards: Grade 4 excerpt

Standard 1 — Number Sense
Understanding the number system is the basis of mathematics. Students extend their understanding of the place value system to count, read, and write whole numbers up to 1,000,000 and decimals to two places. They order and compare whole numbers using the correct symbols for greater than and less than. They extend the concept of fractions to mixed numbers, learning how fractions are related to whole numbers. They also extend their skills with decimals and how they relate to fractions.

Standard 1 (Grade 4)
Number Sense

Students understand the place value of whole numbers and decimals to two decimal places and how whole numbers and decimals relate to simple fractions.*

4.1.1 Read and write whole numbers up to 1,000,000.
Example: Read aloud the number 394,734.

4.1.2 Identify and write whole numbers up to 1,000,000, given a place-value model.
Example: Write the number that has 2 hundred thousands, 7 ten thousands, 4 thousands, 8 hundreds, 6 tens, and 2 ones.

4.1.3 Round whole numbers up to 10,000 to the nearest ten, hundred, and thousand.
Example: Is 7,683 closer to 7,600 or 7,700? Explain your answer.

4.1.4 Order and compare whole numbers using symbols for "less than" (<), "equal to" (=), and "greater than" (>).
Example: Put the correct symbol in 328 __ 142.

4.1.5 Rename and rewrite whole numbers as fractions.
Example: $3 = \frac{6}{2} = \frac{9}{3} = \frac{?}{4} = \frac{?}{5}$.

4.1.6 Name and write mixed numbers, using objects or pictures.
Example: You have 5 whole straws and half a straw. Write the number that represents these objects.

4.1.7 Name and write mixed numbers as improper fractions, using objects or pictures.
Example: Use a picture of 3 rectangles, each divided into 5 equal pieces, to write $2\frac{3}{5}$ as an improper fraction.

4.1.8 Write tenths and hundredths in decimal and fraction notations. Know the fraction and decimal equivalents for halves and fourths (e.g., $\frac{1}{2} = 0.5 = 0.50$, $\frac{7}{4} = 1\frac{3}{4} = 1.75$).
Example: Write $\frac{26}{100}$ and $\frac{23}{4}$ as decimals.

Standard 2
Computation

Students solve problems involving addition, subtraction, multiplication, and division of whole numbers and understand the relationships among these operations. They extend their use and understanding of whole numbers to the addition and subtraction of simple fractions and decimals.

4.2.1 Understand and use standard algorithms* for addition and subtraction.
Example: 45,329 + 6,984 = ?, 36,296 – 12,075 = ?.

4.2.2 Represent as multiplication any situation involving repeated addition.
Example: Each of the 20 students in your physical education class has 3 tennis balls. Find the total number of tennis balls in the class.

4.2.3 Represent as division any situation involving the sharing of objects or the number of groups of shared objects.
Example: Divide 12 cookies equally among 4 students. Divide 12 cookies equally to find out how many people can get 4 cookies. Compare your answers and methods.

4.2.4 Demonstrate mastery of the multiplication tables for numbers between 1 and 10 and of the corresponding division facts.
Example: Know the answers to 9×4 and $35 \div 7$.

4.2.5 Use a standard algorithm to multiply numbers up to 100 by numbers up to 10, using relevant properties of the number system.
Example: $67 \times 3 = ?$.

4.2.6 Use a standard algorithm to divide numbers up to 100 by numbers up to 10 without remainders, using relevant properties of the number system.
Example: $69 \div 3 = ?$.

4.2.7 Understand the special properties of 0 and 1 in multiplication and division.
Example: Know that $73 \times 0 = 0$ and that $42 \div 1 = 42$.

4.2.8 Add and subtract simple fractions with different denominators, using objects or pictures.
Example: Use a picture of a circle divided into 6 equal pieces to find $5/6 - 1/3$.

4.2.9 Add and subtract decimals (to hundredths), using objects or pictures.
Example: Use coins to help you find \$0.43 – \$0.29.

4.2.10 Use a standard algorithm to add and subtract decimals (to hundredths).
Example: 0.74 + 0.80 = ?.

4.2.11 Know and use strategies for estimating results of any whole-number computations.
Example: Your friend says that 45,329 + 6,984 = 5,213. Without solving, explain why you think the answer is wrong.

4.2.12 Use mental arithmetic to add or subtract numbers rounded to hundreds or thousands.
Example: Add 3,000 to 8,000 without using pencil and paper.

* algorithm: a transparent, step-by-step procedure that generalizes beyond numerical situations

[i] U.S. fourth graders use calculators and computers in mathematics class more frequently than do students in most other Third International Math and Science Study (TIMSS) countries. Use of calculators in U.S. fourth-grade mathematics classes is about twice the international average. In the United States, teachers of 39 percent of the students report having students use calculators in their mathematics classes at least once or twice a week compared with the international average of 18 percent. Internationally, the teachers of two-thirds of the TIMSS students report that they never or hardly ever had students use calculators in their mathematics classes compared with the teachers of one-third of U.S. students. In six of the seven nations that outscore the United States in mathematics, teachers of 85 percent or more of the students report that students never use calculators in class. Retrieved on June 25 from http://nces.ed.gov/pubs97/report/97255-2a.asp#i

[ii] Numerous studies of the results of the TIMSS found that American teachers are tackling an ever-wider range of math topics each year compared to teachers in countries with higher math achievement. "In other countries, they might spend a month on a topic while we spend days on a topic," says William Schmidt, the U.S. research coordinator for TIMSS. The "inch-deep" coverage makes it harder for students to remember what they learned. "Then next year, since they've forgotten it all, we have to review it." As a result, extensive time is spent each year on the same basic skills."

[iii] Using primary 6 (EM1/EM2) — the more rigorous level when compared to (EM3). Two tracks (same content, slower pace)