# Hopf ring

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A *Hopf ring* is a (graded) ring object in the category of (graded) cocommutative coalgebras. Such an object consists, first, of a sequence $\{H_i\}$ of abelian group objects in the category. These are better known as commutative Hopf algebras with conjugation. Being in our category they have a coproduct:

$$\psi \quad : \quad H_i \longrightarrow H_i \otimes H_i.$$

Let $\psi(x) = \sum x' \otimes x''$. We denote the product by $*$. The "$*$" product should be thought of as "addition" in our ring as it is the pairing which gives the abelian group structure. For our ring "multiplication" we have

$$\circ \quad : \quad H_i \otimes H_j. \longrightarrow H_{i+j}.$$

As with any ring, there must be a distributive law relating the multiplication and the addition. Chasing diagrams in our category we see that it is:

$$x \circ (y * z) = \sum \pm (x' \circ y) * (x'' \circ z).$$

Hopf rings arise naturally in the study of the $\Omega$-spectra associated with generalized cohomology theories. Any generalized cohomology theory, $G^*(X)$, gives rise to a sequence of spaces, $\{\underline{G}_k\}$ with the property that $G^k(X) \simeq [X, \underline{G}_k]$, the homotopy classes of maps. If $G$ is a multiplicative theory, then $\{\underline{G}_k\}$ is a graded ring object in the homotopy category. If $E$ represents a generalized homology theory and if there is a Künneth isomorphism for the $E$ homology of the spaces in the $\Omega$-spectra for $G$, then the sequence $\{E_*(\underline{G}_*)\}$ becomes a Hopf ring. We can thus use our understanding of generalized homologies to further our understanding of generalized cohomologies by studying their classifying spaces using Hopf rings.

There are a number of Hopf rings which have been computed. Examples are $E_*(\underline{BP}_*)$ and $E_*(\underline{MU}_*)$, $E$ a complex orientable theory, [RW77] (the basic reference for Hopf rings); $E_*(\underline{K(n)}_*)$ and $E_*(\underline{P(n)}_*)$, $E$ a complex orientable theory with $I_n = 0$, [Wil84] and [RW]; $H_*(K(\mathbf{Z}/(p), *))$, [Wil82, §8]; $K(n)_*(-)$ for

Eilenberg–Mac Lane spaces, [RW80]; $K(n)_*(\underline{k(n)}_{\,*})$, [Kra90]; $H_*(\underline{KO})$, [Str92]; and the breakthrough description of $H_*(QS^0, \mathbf{Z}/(2))$ in [Tur], and its sequel for $H_*(QS^*, \mathbf{Z}/(2))$ in [ETW] followed by corresponding results for odd primes in [Li96]. Other references are [HH], [HR], [Kas94], and [KST96].

Hopf rings have a very rich algebraic structure, useful in two distinct ways: discriptive and computational. All of the above examples have their Hopf rings described with just a few generators and relations. The computations are generally carried out using Hopf ring techniques as well.

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