Motivation

- Developing/Understanding Differential and Integral Calculus using infinitely large and small numbers
- Provide easier and more intuitive proofs of results in analysis
Filters

Definition
Let $I$ be a nonempty set. A filter on $I$ is a nonempty collection $F \subseteq P(I)$ of subsets of $I$ such that:

- If $A, B \in F$, then $A \cap B \in F$.
- If $A \in F$ and $A \subseteq B \subseteq I$, then $B \in F$.

$F$ is proper if $\emptyset \notin F$.

Definition
An ultrafilter is a proper filter such that for any $A \subseteq I$, either $A \in F$ or $A^c \in F$. $F^i = \{A \subseteq I : i \in A\}$ is called the principal ultrafilter generated by $i$. 
Filters

Theorem

*Any infinite set has a nonprincipal ultrafilter on it.*

Pf: Zorn’s Lemma/Axiom of Choice.
The Hyperreals

Let \( \mathbb{R}^\mathbb{N} \) be the set of all real sequences on \( \mathbb{N} \), and let \( F \) be a fixed nonprincipal ultrafilter on \( \mathbb{N} \). Define an (equivalence) relation on \( \mathbb{R}^\mathbb{N} \) as follows:

\[
\langle r_n \rangle \equiv \langle s_n \rangle \text{ iff } \{ n \in \mathbb{N} : r_n = s_n \} \in F.
\]

One can check that this is indeed an equivalence relation. We denote the equivalence class of a sequence \( r \in \mathbb{R}^\mathbb{N} \) under \( \equiv \) by \( [r] \).

Then

\[
*\mathbb{R} = \{ [r] : r \in \mathbb{R}^\mathbb{N} \}.
\]

Also, we define

\[
[r] + [s] = [\langle r_n + s_n \rangle]
\]

\[
[r] \ast [s] = [\langle r_n \ast s_n \rangle]
\]
The Hyperreals

We say \([r] = [s]\) iff \(\{n \in \mathbb{N} : r_n = s_n\} \in F.\) \(<\) is defined similarly. A subset \(A\) of \(\mathbb{R}\) can be enlarged to a subset \(*A\) of \(*\mathbb{R}\), where

\([r] \in *A \iff \{n \in \mathbb{N} : r_n \in A\} \in F.\)

Likewise, a function \(f : \mathbb{R} \to \mathbb{R}\) can be extended to \(*f : *\mathbb{R} \to *\mathbb{R}\), where

\[*f([r]) := [\langle f(r_1), f(r_2), \ldots \rangle]*\]
A hyperreal $b$ is called:

- limited iff $|b| < n$ for some $n \in \mathbb{N}$.
- unlimited iff $|b| > n$ for all $n \in \mathbb{N}$.
- infinitesimal iff $|b| < \frac{1}{n}$ for all $n \in \mathbb{N}$.
- appreciable iff $\frac{1}{n} < |b| < n$ for some $n \in \mathbb{N}$.
Transfer Principle

Statement: A defined $\mathcal{L}_\mathbb{R}$ sentence $\phi$ is true iff $^\ast \phi$ is true.

Examples:

\[ \forall x, y \in \mathbb{R}, x < y \Rightarrow \exists q \in \mathbb{Q}(x < q < y). \]

gets transferred to

\[ \forall x, y \in ^\ast \mathbb{R}, x < y \Rightarrow \exists q \in ^\ast \mathbb{Q}(x < q < y). \]

\[ \forall x, y \in \mathbb{R}, \sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y) \]

gets transferred to

\[ \forall x, y \in ^\ast \mathbb{R}, ^\ast \sin(x + y) = ^\ast \sin(x) ^\ast \cos(y) + ^\ast \cos(x) ^\ast \sin(y) \]
We say a hyperreal $b$ is infinitely close to hyperreal $c$ if $b - c$ is infinitesimal and denote this by $b \simeq c$. One can show that $\simeq$ is an equivalence relation. We define

$$\text{hal}(b) = \{c \in {}^\ast \mathbb{R} : b \simeq c\}.$$ 

**Theorem**

*Every limited hyperreal $b$ is infinitely close to exactly one real number, called the shadow of $b$, denoted by $\text{sh}(b)$.*
Note that a real-valued sequence is a function from $\mathbb{N} \to \mathbb{R}$, so it extends to a hypersequence mapping $\ast\mathbb{N} \to \ast\mathbb{R}$.

**Theorem**

A real valued sequence $\langle s_n \rangle$ converges to $L \in \mathbb{R}$ iff $s_n \simeq L$ for all unlimited $n$.

**Theorem**

A real valued sequence $\langle s_n \rangle$ is Cauchy in $\mathbb{R}$ iff for all $m, n$ unlimited hypernaturals, $s_m \simeq s_n$.

Using these concepts, we can prove that a real-valued sequence $s$ convergent in $\mathbb{R}$ $\Rightarrow$ $s$ is Cauchy.
Pf: Suppose $\langle s_n \rangle$ converges in $\mathbb{R}$. Then by the first theorem, $s_n \simeq L$ for all unlimited $n$. So for all $l, m$ unlimited hypernaturals, $s_l \simeq L \simeq s_m \Rightarrow s_l \simeq s_m$ because $\simeq$ an equivalence relation. Then by the second theorem, $\langle s_n \rangle$ is Cauchy.
**Continuity**

**Theorem**

\( f \) is continuous at \( c \in \mathbb{R} \) iff \( \ast f(x) \simeq \ast f(c) \) for all \( x \in \ast \mathbb{R} \) such that \( x \simeq c \).

**Example:** \( f(c) = c^2 \). Let \( c \) be real and \( x \simeq c \). Then \( x = c + \epsilon \) for some infinitesimal \( \epsilon \), and

\[
\begin{align*}
    f(x) - f(c) &= x^2 - c^2 \\
    &= (c + \epsilon)^2 - c^2 \\
    &= c^2 + 2\epsilon c + \epsilon^2 - c^2 \\
    &= 2\epsilon c + \epsilon^2
\end{align*}
\]

which is infinitesimal because \( c \) is a real number and so it is limited. Thus, \( c^2 \) is continuous.
Another Application:

**Theorem**

Let $f$ be a real function defined on some open neighborhood of $c \in \mathbb{R}$, and let $\ast f$ be constant on $\text{hal}(c)$. Then $f$ is constant on some open interval $(c - \epsilon, c + \epsilon) \subseteq \mathbb{R}$.

**Pf:** Note that for some positive infinitesimal $d$, we have the statement $\forall x \in \ast \mathbb{R}$ such that $(|x - c|) < d$, $\ast f(x) = \ast f(c) = L$ for some $L$. This implies that $\exists y \in \ast \mathbb{R}^+$, $\forall x \in \ast \mathbb{R}$ such that $(|x - c|) < y$, $\ast f(x) = \ast f(c) = L \in \ast \mathbb{R}$. By transfer, we have the sentence $\exists y \in \mathbb{R}^+$, $\forall x \in \mathbb{R}$ such that $(|x - c|) < y$, $f(x) = f(c) = L \in \mathbb{R}$. Thus, $f$ is constant on the interval $(c - y, c + y) \subseteq \mathbb{R}$. 
Differentiation

**Theorem**

If $f$ is defined at $x \in \mathbb{R}$, then $L \in \mathbb{R}$ is the derivative of $f$ at $x$ iff for every nonzero infinitesimal $\epsilon$, $^*f(x + \epsilon)$ is defined and

$$\frac{^*f(x + \epsilon) - ^*f(x)}{\epsilon} \simeq L.$$  

**Example:** Consider the real-valued function $\sin(x)$, where $x \in \mathbb{R}$. Now consider

$$\frac{\sin(x + \epsilon) - \sin(x)}{\epsilon}$$

for some $\epsilon$ an infinitesimal. Then by sum of sines, we get

$$\frac{\sin(x + \epsilon) - \sin(x)}{\epsilon} = \frac{\sin(x) \cos(\epsilon) + \cos(x) \sin(\epsilon) - \sin(x)}{\epsilon}$$
\( \cos(x) \) is continuous, so \( \cos(\epsilon) \simeq \cos(0) = 1 \) and so \( \sin(x) \cos(\epsilon) \simeq \sin(x) \). Thus,

\[
\frac{\sin(x) \cos(\epsilon) + \cos(x) \sin(\epsilon) - \sin(x)}{\epsilon} \simeq \frac{\cos(x) \sin(\epsilon)}{\epsilon}
\]

Also, \( \sin(x) \) is continuous, so \( \sin(\epsilon) \simeq \sin(0) = 0 \), so \( \sin(\epsilon) \simeq \epsilon \) and

\[
\frac{\cos(x) \sin(\epsilon)}{\epsilon} \simeq \cos(x).
\]

By the theorem, this implies that the derivative of \( \sin(x) \) at \( x \in \mathbb{R} \) is \( \cos(x) \).
Overview

- The Transfer Principle is key.
- Nonstandard Analysis makes analysis easier!