CLASSIFICATION OF GROUPS WITH ORDER $\leq 20$

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Theorem. (Classification of Finite Abelian Groups)

Let $G$ be a finite Abelian group. Then

$$P \cong C_{d_1} \oplus C_{d_2} \oplus \cdots \oplus C_{d_n}.$$ 

where $d_1, d_2, \ldots, d_n$ are (possibly non-distinct) powers of prime numbers, up to reordering.
Theorem. (Sylow Theorem)

Let $G$ be a finite group and $p$ a prime divisor of $|G|$. Then

1. There exists a $p$-Sylow subgroup in $G$.

2. If $P_1, P_2$ are two $p$-Sylow subgroups in $G$, then for some $g$

   $$P_1 = gP_2g^{-1}.$$ 

3. Let $n$ be the number of $p$-Sylow subgroup in $G$, then

   $$n \equiv 1 \mod p.$$
Definition. (Semidirect product)

Given a group $G$, a subgroup $H$, and a normal subgroup $N$ in $G$:

$G = N \rtimes H$, where $N \cap H = \{e\}$.

The multiplication in $G$ is given by $(a_1, b_1) \cdot_G (a_2, b_2) = (a_1 \theta_{b_1}(a_2), b_1 b_2)$.

where $\theta: H \to \text{Aut}(N)$
1. Locate a normal subgroup in $G$, call it $N$.

2. Try to find another subgroup $H$ in $G$ that has trivial intersection with $N$ such that $|G| = |N||H|$.

3. Then $G = N \rtimes H$. Each possible structure for $N$, $H$, and the action of $H$ on $N$ that defines multiplication in $G$ leads to an unique structure for $G$. 
### RESULT

<table>
<thead>
<tr>
<th>Groups with Prime Orders $p$:</th>
<th>$C_p$</th>
<th>1, 2, 3, 5, 7, 11, 13, 17, 19</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Groups with Orders $p^2$:</th>
<th>$C_{p^2}$</th>
<th>$C_p \times C_p$</th>
<th>4, 9</th>
</tr>
</thead>
</table>

Act on itself using **left multiplication**.

Use the class formula to prove $Z(G)$ is a **nontrivial p-group**.

Use the fact that if $G/Z(G)$ is cyclic then $G$ is Abelian to show $G$ is **Abelian**.

Use the classification theorem.
Groups with **Order** $pq$: $p < q, q \equiv 1 \mod p$

| $C_{pq}$ | $C_p \ltimes C_q$ | 6, 10, 14 |

Use Sylow theorem to show the **q-Sylow subgroup is unique** and thus **normal**.

$$G \cong \text{Syl}(p) \rtimes \text{Syl}(q)$$

Action of Syl$(q)$ on Syl$(p)$ uniquely determined by **action of** $b$ **on** $a$.

*Trivial action* leads to $C_{pq}$.

*Any other action* leads to $C_p \ltimes C_q$ after switching generators.
Groups with **Order** $pq$: $p < q, q \equiv a \mod p, a \neq 1$

Use Sylow theorem to show $\text{Syl}(p)$ and $\text{Syl}(q)$ are normal.

Use their normality to show $\text{Syl}(p)$ commutes with $\text{Syl}(q)$.

$G \cong \text{Syl}(p) \rtimes \text{Syl}(q)$

But since the two groups commute, **the action is trivial**, so the semi product is just a **direct product**.
Use Classification Theorem to handle the Abelian case.

For the non-Abelian case, there must be order 4 element $y$ and another element $x$ not in $\langle y \rangle$.

Conclude the structure based on $x^2$.

\[
x^2 = e \text{ gives } D_8.
\]
\[
x^2 = y^2 \text{ gives } Q_8.
\]
Groups with **Order 16**: $C_{16}$ $C_8 \times C_2$ $C_4 \times C_4$ $C_2 \times C_2 \times C_4$ $C_2 \times C_2 \times C_2 \times C_2$

$D_{16}$ $D_8 \times C_2$ $Q_8 \times C_2$ $C_4 \rtimes C_4$ $D_{ic_4}$ ($C_2 \times C_2 \rtimes C_4$

\{a, b, c|a^4 = b^2 = c^2 = e, ba = ab, ca = ac, cb = a^2bc\}

\{a, b|a^8 = b^2 = e, ba = a^5b\}

\{a, b|a^8 = b^2 = e, ba = a^3b\}

Divides into several cases based on the size and structure of $Z(G)$.

Consider the **correspondence groups in $G$** of **subgroups in $G/Z(G)$**.
## RESULT

Groups with **Order 12**:  
<table>
<thead>
<tr>
<th>Group</th>
<th>Description</th>
<th>Presentation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{12}$</td>
<td>$C_2 \times C_2 \times C_3$</td>
<td></td>
</tr>
<tr>
<td>$D_{12}$</td>
<td>$A_4$</td>
<td>$\langle a, b, c</td>
</tr>
</tbody>
</table>

Groups with **Order 18**:  
<table>
<thead>
<tr>
<th>Group</th>
<th>Description</th>
<th>Presentation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{18}$</td>
<td>$C_3 \times C_3 \times C_2$</td>
<td></td>
</tr>
<tr>
<td>$D_{18}$</td>
<td>$S_3 \rtimes Z_2$</td>
<td>$E_9$</td>
</tr>
</tbody>
</table>

Groups with **Order 20**:  
<table>
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<tr>
<th>Group</th>
<th>Description</th>
<th>Presentation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{20}$</td>
<td>$C_2 \times C_2 \times C_5$</td>
<td></td>
</tr>
<tr>
<td>$D_{20}$</td>
<td>$C_5 \rtimes C_4$</td>
<td>$\langle a, b, c</td>
</tr>
</tbody>
</table>

All three are proved in the same way. I’ll explain groups of order 20 in detail.
We simply need to consider the action $\theta$ of $\text{Syl}(2)$ on $\text{Syl}(5)$.

where $\theta : \text{Syl}(2) \rightarrow \text{Aut}(\text{Syl}(5))$

Note that $\text{Syl}(5) \cong C_5 = \langle a \rangle$ and $\text{Aut}(\text{Syl}(5)) \cong C_4$. 
CASE I: $\text{Syl}(2) \cong C_4 = \langle b \rangle$.

$\theta_1(b)(a) = a^2$: Let $C_4 \cong \text{Syl}(2)$ identify with $C_4$ in $\text{Aut}(\text{Syl}(5))$.
leads to $C_5 \rtimes C_4$

$\theta_2(b)(a) = a^4$: Send $C_4 \cong \text{Syl}(2)$ to $C_2$ in $\text{Aut}(\text{Syl}(5))$.
Identify $x = (a, b^2), y = (a, b), z = (e, b)$.
leads to $\langle a, b, c | a^5 = b^2 = c^2 = abc \rangle$

$\theta_3(b)(a) = a$: Trivial action.
leads to $C_{20}$
Case II: $\text{Syl}(2) \cong C_2 \times C_2 = \{e, b, c, bc\}$.

$\theta_1(b)(a) = a^4$: Let two $C_2$ in $\text{Syl}(2)$ identify with $C_2$ in $\text{Aut}(\text{Syl}(5))$.
$\theta_1(c)(a) = a^4$ Identify $x = (a, b)$ and $y = (a^3, bc)$.
leads to $D_{20}$

$\theta_2(b)(a) = a$: Trivial action.
$\theta_2(c)(a) = a$
leads to $C_5 \times C_2 \times C_2$

We have finished our classification.