ALGEBRA CORE QUALIFYING EXAM. SEPTEMBER, 2001.

Directions: Solve four problems from the following list of five and clearly indicate which problems you chose as only those will be graded. Show all your work.

In general, it is permissible to use earlier parts of a problem in order to solve a later part even if you have not solved the earlier parts.

1. Let G be a finite group and let N be a normal subgroup of G such that N and G/N have relatively prime orders.
(a) Assume that there exists a subgroup H of G having the same order as G/N. Show that $G = HN$. (Here HN denotes the set $\{xy \mid x \in H,\, y \in N\}$.)
(b) Show that $\phi(N) = N$, for all automorphisms ϕ of G.

2. Let S denote the ring $\mathbb{Z}[X]/X^2\mathbb{Z}[X]$, where X is a variable.
(a) Show that S is a free $\mathbb{Z}$–module and find a $\mathbb{Z}$–basis for S.
(b) Which elements of S are units (i.e. invertible with respect to multiplication) ?
(c) List all the ideals of S.
(d) Find all the nontrivial ring morphisms defined on S and taking values in the ring of Gaussian integers $\mathbb{Z}[i]$.

3. Assume that A is an $n \times n$ matrix with entries in the field of complex numbers $\mathbb{C}$ and $A^m = 0$ for some integer $m > 0$.
(a) Show that if λ is an eigenvalue of A, then $\lambda = 0$.
(b) Determine the characteristic polynomial of A.
(c) Prove that $A^n = 0$.
(d) Write down a 5×5 matrix B for which $B^3 = 0$ but $B^2 \neq 0$.
(e) If M is any 5×5 matrix over $\mathbb{C}$ with $M^3 = 0$ and $M^2 \neq 0$, must M be similar to the matrix B you found in part (d) ? Justify your answer.

4. Let $K := \mathbb{Q}(\sqrt{3} + \sqrt{5})$.
(a) Show that K is the splitting field of $X^4 - 6X^2 + 4$.
(b) Find the structure of the Galois group of $K/\mathbb{Q}$.
(c) List all the fields k, satisfying $\mathbb{Q} \subseteq k \subseteq K$.

5. Let $\rho : G \longrightarrow \mathrm{Gl}_n(\mathbb{C})$ be a complex irreducible representation of degree n of a finite group G. Let χ be its associated character and let C be the center of G.
(a) Show that, for all $s \in C$, $\rho(s)$ is a scalar multiple of the identity matrix I_n.
(b) Use (a) to show that $|\chi(s)| = n$, for all $s \in C$.
(c) Prove the inequality $n^2 \leq [G : C]$, where $[G : C]$ denotes the index of C in G.
(d) Show that, if ρ is faithful (i.e. an injective group morphism), then the group C has to be cyclic.