Algebra Core Qualifying Exam, September, 2002

Notation: Integers: $\mathbb{Z}$, the integers modulo p: $\mathbb{Z}/(p)$, rationals: $\mathbb{Q}$.

(1) (3 points) Let H be a subgroup of G with $a, b \in G$. Prove (without assuming very much) that the right cosets Ha and Hb are either equal or disjoint.

(2) (3 points) Prove that if H is a subgroup of index 2 in G then H is normal in G.

(3) (3 points) Working over the integers, calculate (and show your work in a readable fashion) $\text{Tor}(\mathbb{Z}/(p), \mathbb{Z}/(p))$.

(4) (3 points) Working over the integers, calculate (and show your work in a readable fashion) $\text{Ext}(\mathbb{Z}/(p), \mathbb{Z}/(p))$.

(5) (6 points) Recall $D_4 = \{1, a, a^2, a^3, ba, ba^2, ba^3\}$, $|a| = 4$, $|b| = 2$, $aba = b$. Find the center of D_4, $Z(D_4)$, and describe $D_4/Z(D_4)$.

(6) (3 points) Calculate all the group homomorphisms from S_3, the symmetric group on 3 elements, to $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$. Explain your answer.

(7) (1 point) How many monic polynomials of degree 3 are there over $\mathbb{Z}/(3)$?

(8) (3 points) How many irreducible monic polynomials of degree 3 are there over $\mathbb{Z}/(3)$.

(9) (5 points) For every monic polynomial, $f(x)$, in problem # 7, we can define the quotient ring $\mathbb{Z}/(3)[x]/(f(x))$. How many different rings do we get if we use only the monic polynomials of problem # 8? Explain your answer and identify your answers as familiar rings.

(10) (3 points) Find all the idempotents not equal to 0 or 1 in the ring $\mathbb{Z}/(2)[x]/(x^3 + 1)$.

(11) (4 points) Find the minimal polynomial for $\sqrt{2} + i$ over $\mathbb{Q}$.

(12) (10 points) Demonstrate your knowledge of Galois theory for the field extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2} + i)$ (in the complex numbers).

(13) (3 points) Prove Cauchy's theorem that if a prime p divides the order of a group, $|G|$, then G has an element of order p. (You can assume the result for Abelian groups.)

(14) (3 points) Give all groups of order 175. Two are pretty easy, it is the rest I care about. Explain your answer.