Qualifying Exam in Algebra, Fall 2003

Directions: This is a closed book exam. You have two hours to do all five of the (equally weighted) problems.

1. In a group G, let 1 denote the identity element and let $[x, y] = xyx^{-1}y^{-1}$ denote the commutator of the elements $x, y \in G$.

a) Express $[z, xy]x$ in terms of x, $[z, x]$ and $[z, y]$.

b) Prove that if the identity $[[x, y], z] = 1$ holds in a group G, then the identities

$$[x, yz] = [x, y][x, z] \quad \text{and} \quad [xy, z] = [x, z][y, z] \quad \text{hold in } G.$$

2. Let k be a field of characteristic p and let t, u be algebraically independent over k. Prove the following:

a) $k(t, u)$ has degree p^2 over $k(t^p, u^p)$.

b) There exist infinitely many fields between $k(t, u)$ and $k(t^p, u^p)$.

3. Obtain a factorization into irreducible factors in $\mathbb{Z}[x]$ of the polynomial $x^{10} - 1$.

4. Verify the isomorphism of algebras over a field K:

$$\mathbb{M}_n(K) \otimes_K \mathbb{M}_m(K) \simeq \mathbb{M}_{mn}(K).$$

[Note: $\mathbb{M}_n(K)$ denotes the algebra of $n \times n$ matrices over K.]

5. Let $\mathbb{S}_4$ be the *symmetric* group on 4 elements.

a) Give an example of a non-trivial 8-dimensional representation of the group $\mathbb{S}_4$.

b) Prove for any 8-dimensional complex representation of $\mathbb{S}_4$ the existence of a 2-dimensional invariant subspace.