Do all problems. All problems are equally weighted. Show all details.

1. Let $H$ be a proper subgroup of a finite group $G$. Show that $G$ is not the union of all the conjugates of $H$.

2. Let $\mathfrak{N}$ be the set of all nilpotent elements in a ring $R$. Assume first that $R$ is commutative.
   (a) Show that $\mathfrak{N}$ is an ideal in $R$, and $R/\mathfrak{N}$ contains no non-zero nilpotent elements.
   (b) Show that $\mathfrak{N}$ is the intersection of all the prime ideals of $R$.
   (c) Give an example with $R$ non-commutative where $\mathfrak{N}$ is not an ideal in $R$.

3. Let $f(x) = x^5 - 9x + 3$. Determine the Galois group of $f$ over $\mathbb{Q}$.

4. Let $\lambda_1, \ldots, \lambda_n$ be roots of unity, with $n \geq 2$. Assume that $\frac{1}{n} \sum_{i=1}^{n} \lambda_i$ is integral over $\mathbb{Z}$. Show that either $\sum_{i=1}^{n} \lambda_i = 0$ or $\lambda_1 = \lambda_2 = \cdots = \lambda_n$.

5. Consider the ideal $I = (2, x)$ in $R = \mathbb{Z}[x]$.
   (a) Construct a non-trivial $R$-module homomorphism $I \otimes_R I \to R/I$, and use that to show that $2 \otimes x - x \otimes 2$ is a non-zero element in $I \otimes_R I$.
   (b) Determine the annihilator of $2 \otimes x - x \otimes 2$.

6. Let $D_8$ be the dihedral group of order 8, given by generators and relations
   \[
   < r, s \mid r^4 = 1 = s^2, \ rs = sr^{-1} >
   \]
   (a) Determine the conjugacy classes of $D_8$.
   (b) Determine the commutator subgroup $D'_8$ of $D_8$. Determine the number of distinct degree one characters of $D_8$.
   (c) Write down the complete character table of $D_8$. 