Algebra Qualifying Exam Fall 2004

September 7, 2004

Do all problems. All problems are equally weighted. Show all details.

1. Let H be a proper subgroup of a finite group G. Show that G is not the union of all the conjugates of H.

2. Let $\mathfrak{N}$ be the set of all nilpotent elements in a ring R. Assume first that R is commutative.

(a) Show that $\mathfrak{N}$ is an ideal in R, and $R/\mathfrak{N}$ contains no non-zero nilpotent elements.

(b) Show that $\mathfrak{N}$ is the intersection of all the prime ideals of R.

(c) Give an example with R **non**-commutative where $\mathfrak{N}$ is not an ideal in R.

3. Let $f(x) = x^5 - 9x + 3$. Determine the Galois group of f over $\mathbb{Q}$.

4. Let $\lambda_1, \ldots, \lambda_n$ be roots of unity, with $n \geq 2$. Assume that $\frac{1}{n}\sum_{i=1}^{n} \lambda_i$ is integral over $\mathbb{Z}$. Show that either $\sum_{i=1}^{n} \lambda_i = 0$ or $\lambda_1 = \lambda_2 = \cdots = \lambda_n$.

5. Consider the ideal $I = (2, x)$ in $R = \mathbb{Z}[x]$.

(a) Construct a non-trivial R-module homomorphism $I \otimes_R I \to R/I$, and use that to show that $2 \otimes x - x \otimes 2$ is a non-zero element in $I \otimes_R I$.

(b) Determine the annihilator of $2 \otimes x - x \otimes 2$.

6. Let D_8 be the dihedral group of order 8, given by generators and relations

$$< r, s \,|\, r^4 = 1 = s^2,\ rs = sr^{-1} >$$

(a) Determine the conjugacy classes of D_8.

(b) Determine the commutator subgroup D_8' of D_8. Determine the number of distinct degree one characters of D_8.

(c) Write down the complete character table of D_8.