Algebra Qualifying Exam Fall 2005

September 20, 2005 (150 minutes)

Do all problems. All problems are equally weighted. Show all work.

1. Let K be a finite field with q elements. Let $n > 0$ be a positive integer. Compute the sum

$$\sum_{x \in K} x^n.$$

2. Let K be the splitting field (in $\mathbb{C}$) of the polynomial $x^4 - 3x^2 + 5$ over $\mathbb{Q}$.
(1). Determine $\mathrm{Gal}(K/\mathbb{Q})$.
(2). Find all intermediate fields $\mathbb{Q} \subset E \subset K$ such that E is Galois over $\mathbb{Q}$.

3. Let $k \subset E$ be an algebraic extension of fields of characteristic zero. Assume that every non-constant polynomial $f(x) \in k[x]$ has a root in E. Show that E is algebraically closed.

4. Let R be a commutative ring. Let I be a finitely generated ideal. Assume that $I^2 = I$. Show that I is a direct summand of R.

5. Let $\mathbb{C}$ and $\mathbb{R}$ be complex and real number fields. Let $\mathbb{C}(x)$ and $\mathbb{C}(y)$ be function fields of one variable. Consider $\mathbb{C}(x) \otimes_{\mathbb{R}} \mathbb{C}(y)$ and $\mathbb{C}(x) \otimes_{\mathbb{C}} \mathbb{C}(y)$.
(1). Determine if they are integral domains.
(2). Determine if they are fields.

6. Let E be a finite-dimensional vector space over a field k. Assume $S, T \in \mathrm{End}_k(E)$. Assume $ST = TS$ and both of them are diagonalizable. Show that there exists a basis of E consisting of eigenvectors for both S and T.