Algebra Qualifying Exam Fall 2006

September 13, 2006 (150 minutes)

Do all problems. All problems are equally weighted. Show all work.

1. Let $SL_n(k)$ be the special linear group over a field k, i.e, $n \times n$ matrices with determinant 1. Let I be the identity matrix, and E_{ij} be the elementary matrix that has 1 at (i,j)-entry and 0 elsewhere. Here $1 \leq i \neq j \leq n$.

(1). Let C_{ij} be the centralizer of the matrix $I + E_{ij}$. Find explicit generators of C_{ij}.

(2). Find the intersection

$$\bigcap_{1 \leq i \neq j \leq n} C_{ij}.$$

(3). Determine all the elements in the conjugacy class of $I + E_{ij}$.

2. Let f be a polynomial in $\mathbb{Q}[x]$. Let E be a splitting field of f over $\mathbb{Q}$. For the following cases, determine whether E is solvable by radicals.

(1). $f(x) = x^4 - 4x + 2$.

(2). $f(x) = x^5 - 4x + 2$.

3. Let A be a principal integral domain and K be its field of fractions. Assume that R is a ring such that $A \subset R \subset K$. Show that R is also a principal integral domain.

4. Let R be a commutative ring. Let M be an R-module.

(1). Write down the definition of $\mathcal{T}(M)$, the tensor algebra of M.

(2). Assume $R = \mathbb{Z}$ and $M = \mathbb{Q}/\mathbb{Z}$. Compute $\mathcal{T}(M)$.

(3). If M is a vector space over a field R, show that $\mathcal{T}(M)$ contains no zero divisors.

5. Let A be an invertible $n \times n$ matrix over complex numbers $\mathbb{C}$. Show that A has a square root, i.e, there exists a $n \times n$ matrix B such that $A = B^2$.

6. Let R be a semi-simple finite dimensional algebra over complex numbers $\mathbb{C}$. Let M be an R-module such that $M = E \oplus E$. Here E is a simple R-module. Show that $\mathrm{Aut}_R(M) \cong GL_2(\mathbb{C})$.