Algebra Qualifying Exam, Fall, 2007

Sept 11, 2007, 2:00-4:30

1. Let G be a cyclic group of order 12. Construct a Galois extension K over $\mathbf{Q}$ so that the Galois group is isomorphic to G.

2. Prove that no group of order 148 is simple.

3. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a real matrix such that $a, b, c, d > 0$. (1) Prove that A has two distinct real eigenvalues, $\lambda > \mu$. (2) Prove that λ has an eigenvector in the first quadrant and μ has an eigenvector in the second quadrant.

4. A *differentiation* of a ring R is a mapping $D : R \to R$ such that, for all $x, y \in R$,

   (1) $D(x + y) = D(x) + D(y)$; and

   (2) $D(xy) = D(x)y + xD(y)$.

   If K is a field and R is a K-algebra, then its differentiation are supposed to be over K, that is,

   (3) $D(x) = 0$ for any $x \in K$.

   Let D be a differentiation of the K-algebra $M_n(K)$ of $n \times n$-matrices. Prove that there exists a matrix $A \in M_n(K)$ such that $D(X) = AX - XA$ for all $X \in M_n(K)$.

5. Prove the existence of a 1-dimensional invariant subspace for any 5-dimensional representation of the group A_4 (the *alternating* group of degree 4).