Algebra Qualifying Exam Fall 2008

September 10, 2008 (150 minutes)

Do all problems. All problems are equally weighted. Show all work.

1. Show that no group of order 36 is simple.

2. Show that the polynomial $x^5 - 5x^4 - 6x - 2$ is irreducible in $\mathbb{Q}[x]$.

3. Let k be a finite field and K be a finite extension of k. Let $\mathfrak{Tr} = Tr_k^K$ be the trace function from K to k. Determine the image of $\mathfrak{Tr}$ and prove your answer.

4. A differentiation of a ring R is a mapping $D : R \to R$ such that, for all $x, y \in R$,
(1) $D(x + y) = D(x) + D(y)$; and
(2) $D(xy) = D(x)y + xD(y)$.
If K is a field and R is a K-algebra, then its differentiation are supposed to be over K, that is,
(3) $D(x) = 0$ for any $x \in K$.
Let D be a differentiation of the K-algebra $M_n(K)$ of $n \times n$-matrices. Prove that there exists a matrix $A \in M_n(K)$ such that $D(X) = AX - XA$ for all $X \in M_n(K)$.

5. For each $n \in \mathbb{Z}$, define the ring homomorphism

$$\phi_n : \mathbb{Z}[x] \to \mathbb{Z} \text{ by } \phi_n(f) = f(n).$$

This gives a $\mathbb{Z}[x]$-module structure on $\mathbb{Z}$, i.e,

$$f \circ a = f(n) \cdot a \text{ for all } f \in \mathbb{Z}[x] \text{ and } a \in \mathbb{Z}.$$

Now given two integers $m, n \in \mathbb{Z}$, compute the tensor product $\mathbb{Z} \otimes_{\mathbb{Z}[x]} \mathbb{Z}$ where the left-hand copy of $\mathbb{Z}$ uses the module structure from ϕ_n and the right-hand copy of $\mathbb{Z}$ uses the module structure from ϕ_m. (Note: The answer depends on the numbers n and m.)

6. Let R be a semi-simple finite dimensional algebra over complex numbers $\mathbb{C}$. Let M be an R-module such that $M = E \oplus E$. Here E is a simple R-module. Show that $\mathrm{Aut}_R(M) \cong GL_2(\mathbb{C})$.