Algebra Qualifying Exam Fall 2009

September 9, 2009 (150 minutes)

Do all problems. All problems are equally weighted. Show all work.

1. Let G be a finite group. Let $Aut(G)$ be the group of automorphisms of G. Consider the group action $\phi : Aut(G) \times G \to G$ where $\phi(\sigma, g) = \sigma(g)$. Assume G has exactly two orbits under the action of $Aut(G)$.

(a). Determine all such G, up to isomorphism.

(b). List all cases in which $Aut(G)$ is a solvable group.

2. Consider $\mathbb{Q}[\sqrt{5}] = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$. Determine the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{5}]$.

3. Determine the Galois group of $x^4 - 4x^2 + 7x - 3$ over $\mathbb{Q}$.

4. Let E and F be finite field extensions of a field k such that $E \cap F = k$, and that E and F are both contained in a larger field L. Assume that E is Galois over k. Show that $E \otimes_k F \cong EF$.

5. Let A, B be two Noetherian local rings with maxima ideals m_A, m_B, respectively. Let $f : A \to B$ be a ring homomorphism such that $f^{-1}(m_B) = m_A$. Assume that:

1. $A/m_A \to B/m_B$ is an isomorphism.
2. $m_A \to m_B/m_B^2$ is surjective.
3. B is a finitely generated A-module (via f).

Show that f is surjective.

6. Let $\rho : G \to Gl_n(\mathbb{C})$ be a complex irreducible representation of a finite group G. Let χ be its associated character and let C be the center of G.

(a). Show that, for all $s \in C$, $\rho(s)$ is a scalar multiple of the identity matrix I_n.

(b). Use (a) to show that $|\chi(s)| = n$, for all $s \in C$.

(c). Prove the inequality $n^2 \leq [G : C]$, where $[G : C]$ denotes the index of C in G.

(d). Show that, if ρ is faithful (i.e., an injective group homomorphism), then C is cyclic.