Algebra Qualifying Exam  Fall 2010

September 6, 2010  (150 minutes)

Do all problems. All problems are equally weighted. Show all work.

1. Let $G$ be a group. Let $H$ be a subset of $G$ that is closed under group multiplication. Assume that $g^2 \in H$ for all $g \in G$. Show that $H$ is a normal subgroup of $G$ and $G/H$ is abelian.

2. (a) Find the complete factorization of the polynomial $f(x) = x^6 - 17x^4 + 80x^2 - 100$ in $\mathbb{Z}[x]$.
   
   (b) For which prime numbers $p$ does $f(x)$ have a root in $\mathbb{Z}/p\mathbb{Z}$ (i.e, $f(x)$ has a root modulo $p$)? Explain your answer.

3. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{-1})$ and $F = \mathbb{Q}(\sqrt{-2})$. Show that $K$ is Galois over $F$ and determine the Galois group $\text{Gal}(K/F)$.

4. Let $A$ be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$. Assume $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some $n > 0$. Show that $A$ is Artinian.

5. Let $\mathbb{F}_q$ be a finite field with $q = p^n$ elements. Here $p$ is a prime number. Let $\varphi : \mathbb{F}_q \to \mathbb{F}_q$ be given by $\varphi(x) = x^p$.
   
   (a) Show that $\varphi$ is a linear transformation on $\mathbb{F}_q$ (as vector space over $\mathbb{F}_p$), then determine its minimal polynomial.
   
   (b) Suppose that $\varphi$ is diagonalizable over $\mathbb{F}_p$. Show that $n$ divides $p - 1$.

6. Let $G$ be a non-abelian group of order $p^3$. Here $p$ is a prime number.
   
   (a) Determine the number of (isomorphic classes of) irreducible complex representations of $G$, and find their dimensions.
   
   (b) Which of the irreducible complex representations of $G$ are faithful? Explain your answer.