Algebra Qualifying Exam   Fall 2010

September 6, 2010   (150 minutes)

Do all problems. All problems are equally weighted. Show all work.

1. Let G be a group. Let H be a subset of G that is closed under group multiplication. Assume that $g^2 \in H$ for all $g \in G$. Show that H is a normal subgroup of G and G/H is abelian.

2. (a) Find the complete factorization of the polynomial $f(x) = x^6 - 17x^4 + 80x^2 - 100$ in $\mathbb{Z}[x]$.

(b) For which prime numbers p does $f(x)$ have a root in $\mathbb{Z}/p\mathbb{Z}$ (i.e, $f(x)$ has a root modulo p)? Explain your answer.

3. Let $K = \mathbb{Q}(\sqrt[8]{2}, \sqrt{-1})$ and $F = \mathbb{Q}(\sqrt{-2})$. Show that K is Galois over F and determine the Galois group $\mathrm{Gal}(K/F)$.

4. Let A be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$. Assume $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some $n > 0$. Show that A is Artinian.

5. Let $\mathbb{F}_q$ be a finite field with $q = p^n$ elements. Here p is a prime number. Let $\varphi : \mathbb{F}_q \to \mathbb{F}_q$ be given by $\varphi(x) = x^p$.

(a) Show that φ is a linear transformation on $\mathbb{F}_q$ (as vector space over $\mathbb{F}_p$), then determine its minimal polynomial.

(b) Supposed that φ is diagonalizable over $\mathbb{F}_p$. Show that n divides $p - 1$.

6. Let G be a non-abelian group of order p^3. Here p is a prime number.

(a) Determine the number of (isomorphic classes of) irreducible complex representations of G, and find their dimensions.

(b) Which of the irreducible complex representations of G are faithful? Explain your answer.