Algebra Qualifying Exam, September 2011

All problems are equally weighted. Explain clearly how you arrive at your solutions. You will be given three hours in which to complete the exam.

1. a) Let G be a group of order 5046. Show that G cannot be a simple group. *You may not appeal to the classification of finite simple groups.*

   b) Let p and q be prime numbers. Show that any group of order p^2q is solvable.

2. Consider the special orthogonal group $G = SO(3, \mathbb{R})$, namely,

   $$G = \{A \in GL(3, \mathbb{R}) : \ A^T A = \mathrm{I}_3, \ det(A) = 1\}$$

   (a) Show that for any element A in G, there exists a real number α with $-1 \leq \alpha \leq 3$ such that

   $$A^3 - \alpha A^2 + \alpha A - \mathrm{I}_3 = 0.$$

   (b) For which real numbers α with $-1 \leq \alpha \leq 3$ does there exist an element A in G whose minimal polynomial is $x^3 - \alpha x^2 + \alpha x - 1$? *Explain your answer.*

3. Let G be a cyclic group of order 100. Let $K = \mathbb{Q}$, the field of rational numbers, or $K = F_p$, the finite field with p elements, p being a prime number. For each such K, construct a Galois extension L/K whose Galois group $Gal(L/K)$ is isomorphic to G. *Explain your construction in detail.*

4. Let $\rho : S_3 \to \mathbb{C}^2$ be a two-dimensional irreducible representation of the symmetric group S_3. Decompose $\rho^{\otimes 2}$ and $\rho^{\otimes 3}$ into a direct sum of irreducible representations of S_3.

5. Let A be a finite-dimensional semisimple algebra over $\mathbb{C}$, and V an A-module of finite type (i.e., finitely-generated as an A-module). Prove that V has only finitely many A-submodules if and only if V is a direct sum of pairwise non-isomorphic irreducible (i.e., simple) A-modules.