Algebra Qualifying Exam
Fall 2013

Student's Name:

To receive full credit, show all of your work.

1. Let $p > 2$ be a prime. Classify groups of order p^3 up to isomorphism.

2. Let a be an integral algebraic number such that its norm is 1 for any imbedding into $\mathbb{C}$, the field of complex numbers. Prove that a is a root of unity.

3. Let R be a commutative ring with unity. Given an R-module A and an ideal $I \subset R$, there is a natural R-module homomorphism $A \otimes_R I \to A \otimes_R R \simeq A$ induced by the inclusion $I \subset R$. In the following three steps you shall prove the flatness criterion: *A is flat if and only if for every finitely generated ideal $I \subset R$ the natural map $A \otimes_R I \to A \otimes_R R$ is injective.*

   (a) Prove that if A is flat and $I \subset R$ is a finitely generated ideal then $A \otimes_R I \to A \otimes_R R$ is injective.

   (b) If $A \otimes_R I \to A \otimes_R R$ is injective for every finitely generated ideal I, prove that $A \otimes_R I \to A \otimes_R R$ is injective for every ideal I. Show that if K is any submodule of a free module F then the natural map $A \otimes_R K \to A \otimes_R F \simeq A$ induced by the inclusion $K \subset F$ is injective (*Hint*: the general case reduces to the case when F has finite rank).

   (c) Let $\psi\colon L \to M$ be an injective homomorphism of R-modules. Prove that the induced map $1 \otimes \psi\colon A \otimes_R L \to A \otimes_R M$ is injective (*Hint*: Write M as a quotient $f\colon F \to M$ of a free module F, giving a short exact sequence $0 \to K \to F \to M \to 0$ and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & J & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & K & \longrightarrow & F & \xrightarrow{f} & M & \longrightarrow & 0, \end{array}$$

   where $J = f^{-1}(\psi(L))$.)

4. (a) Let R be a P.I.D. Prove that a finitely generated R-module M is flat if and only if M is torsion-free (hence, free by the structure theorem).

   (b) Give an example of an integral domain R and a torsion-free R-module M such that M is not free.

5. Compute the Galois group of $f(x) = x^4 + 1$ over $\mathbb{Q}$.

6. Let p be a prime and let F be a field of characteristic p.

   (a) Prove that the map $\varphi\colon F \to F, \varphi(a) = a^p$ is a field homomorphism.

   (b) F is said to be *perfect* if the above homomorphism φ is an automorphism. Prove that every finite field is perfect.

   (c) If x is an indeterminate and F is any field of characteristic p, prove that the field $F(x)$ is not perfect.