Algebra Qualifying Exam   Sept 3, 2014

All five problems are equally weighted. (The problem parts need not be equally weighted.) Explain clearly how you arrive at your solutions, or you risk losing credit. You will be given three hours in which to complete the exam.

1. (a) Let S_n be the symmetric group (permutation group) on n objects. Prove that if $\sigma \in S_n$ is an n-cycle and $\tau \in S_n$ is a transposition (i.e., a 2-cycle), then σ and τ generate S_n.

   (b) Let $f_a(x)$ be the polynomial $x^5 - 5x^3 + a$. Determine an integer a with $-4 \leq a \leq 4$ for which f_a is irreducible over $\mathbb{Q}$, and the Galois group of [the splitting field of] f_a over $\mathbb{Q}$ is S_5. Then explain why the equation $f_a(x) = 0$ is not solvable in radicals.

2. Let $R = \mathbb{Q}[X]$, I and J the principal ideals generated by $X^2 - 1$ and $X^3 - 1$ respectively. Let $M = R/I$ and $N = R/J$. Express in simplest terms [the isomorphism type of] the R-modules $M \otimes_R N$ and $\mathrm{Hom}_R(M, N)$. **Explain.**

3. Let $G = S_3$.

   (a) Prove that G has an irreducible complex representation of dimension 2,—call it ρ— but none of higher dimension.

   (b) Decompose $\rho \otimes \rho \otimes \rho$ (as a representation of G) into a direct sum of irreducible representations.

4. (a) Let G be a group of order p^2q^2, where p and q are distinct odd primes, with $p > q$. Show that G has a normal subgroup of order p^2.

   (b) Can a solvable group contain a non-solvable subgroup? **Explain.**

5. (a) Prove that every group of order p^2 (p a prime) is abelian. Then classify such groups up to isomorphism.

   (b) Give an example of a non-abelian group of order p^3 for $p = 3$. *Suggestion: Represent the group as a group of matrices.*