Algebra Qualifying Exam   Sept 2, 2015

All five problems are equally weighted. (The problem parts need not be equally weighted.) Explain clearly how you arrive at your solutions, or you risk losing credit. You will be given three hours in which to complete the exam.

1. Prove that every group of order 15 is cyclic.

2. The dihedral group D_{2n} is the group on two generators r and s, with respective orders $o(r) = n$ and $o(s) = 2$, subject to the relation $rsr = s$.

   (a) Calculate the order of D_{2n}.

   (b) Let K be the splitting field of the polynomial $x^8 - 2$. Determine whether the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ is dihedral (i.e., isomorphic to D_{2n} for some n).

3. Let $G = S_4$ (the symmetric group on four letters).

   (a) Prove that G has two non-equivalent irreducible complex representations of dimension 3; call them ρ_1 and ρ_2.

   (b) Decompose $\rho_1 \otimes \rho_2$ (as a representation of G) into a direct sum of irreducible representations.

4. Let $H = S_3 \times S_5$.

   (a) Determine all normal subgroups of H. *Make sure you have them all!* What would be different if H were replaced by $S_2 \times S_5$?

   (b) Describe, <u>in full detail</u>, the construction of a polynomial with rational coefficients, whose Galois group is isomorphic to H.

5. Let L be a finite field. Let a and b be elements of $L^\times$ (the multiplicative group of L) and $c \in L$. Show that there exist x and y in L such that $ax^2 + by^2 = c$.

6. Let K be a finite algebraic extension of $\mathbb{Q}$.

   (a) Give the definition of an integral element of K.

   (b) Show that the set of integral elements in K form a sub-ring of K.

(c) Determine the ring of integers in each of the following two fields *No credit for memorized answers*: $\mathbb{Q}(\sqrt{13})$, and $\mathbb{Q}(\sqrt[3]{2})$.