Algebra Qualifying Exam, Fall 2016

September 7th, 12:30-3:30

All six problems are equally weighted. (The problem parts need not be equally weighted.) Explain clearly how you arrive at your solutions, or you risk losing credit.

1. Determine $Aut(S_3)$.

2. A group G is a semidirect product of subgroups $N, H \subset G$ if N is normal and every element of G has a unique presentation nh, $n \in N$, $h \in H$. Find all semidirect products (up to isomorphism) of $N = \mathbb{Z}/11\mathbb{Z}$, $H = \mathbb{Z}/5\mathbb{Z}$.

3. Let F be a finite field of order 2^n. Here $n > 0$. Determine all values of n such that the polynomial $x^2 - x + 1$ is irreducible in $F[x]$.

4. (1). Determine the Galois group of $x^4 - 4x^2 - 2$ over $\mathbb{Q}$.

   (2). Let G be a group of order 8 such that G is the Galois group of a polynomial of degree 4 over $\mathbb{Q}$. Show that G is isomorphic to the Galois group in part (1).

5. Let A be a linear transformation of a finite dimensional vector space over a field of characteristic $\neq 2$.

   (1). Define the wedge product linear transformation $\wedge^2 A = A \wedge A$.

   (2). Prove that

   $$tr(\wedge^2 A) = \frac{1}{2}(\, tr(A)^2 - tr(A^2)\,)\,.$$

6. Find a table of characters for the alternating group A_5.