JOHNS HOPKINS UNIVERSITY
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAMINATION
SEPTEMBER 6, 2017, 12:30–3:30 PM

Each problem is worth 10 points.

(1) Show that there is no simple group of order 30.

(2) Let Λ be a free abelian group of finite rank n, and let $\Lambda' \subset \Lambda$ be a subgroup of the same rank. Let $x_1, \ldots, x_n$ be a $\mathbb{Z}$-basis for Λ, and let $x_1', \ldots, x_n'$ be a $\mathbb{Z}$-basis for Λ'. For each i, write $x_i' = \sum_{j=1}^n a_{ij} x_j$, and let $A := (a_{ij}) \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$. Show that the index $[\Lambda : \Lambda']$ equals $|\det A|$.

(3) In this problem all rings are commutative.
   (a) Let R be a local ring with maximal ideal $\mathfrak{m}$, let N and M be finitely generated R-modules, and let $f\colon N \to M$ be an R-linear map. Show that f is surjective if and only if the induced map $N/\mathfrak{m}N \to M/\mathfrak{m}M$ is.
   (b) Recall that a module M over a ring R is *projective* if the functor $\mathrm{Hom}_R(M, -)$ is exact. Show that if R is local and M is finitely generated projective, then M is free.

(4) Compute the Galois group of $x^5 - 10x + 5$ over $\mathbb{Q}$.

(5) Let K/k be an extension of finite fields with $\#k = q$, let $\Phi\colon x \mapsto x^q$ denote the qth power Frobenius map on K, and let $G := \mathrm{Gal}(K/k)$.
   (a) Compute the minimal polynomial of Φ as a k-linear endomorphism of K.
   (b) Use (a) to prove the *normal basis theorem* in the case of the extension K/k: there exists $x \in K$ such that the set $\{\sigma x \mid \sigma \in G\}$ is a k-basis for K.[1] (According to taste, it may be helpful to note that this is equivalent to the statement that $K \simeq k[G]$ as $k[G]$-modules.)

(6) Let G be a finite group with center $Z \subset G$. Show that if G admits a faithful irreducible representation $G \to \mathrm{GL}_n(k)$ for some positive integer n and some field k, then Z is cyclic.

[1] In fact, the normal basis theorem holds for an arbitrary finite Galois field extension K/k, but a different proof is required when k is infinite.