ALGEBRA CORE QUALIFYING EXAM. MARCH, 2001.

Directions: Solve five problems from the following list of six and clearly indicate which problems you chose as only those will be graded. Show all your work.

In general, it is permissible to use earlier parts of a problem in order to solve a later part even if you have not solved the earlier parts.

1. Let G be a finite group and p the smallest prime number dividing the cardinality $|G|$ of G. Let H be a subgroup of G of index p in G. Show that H is necessarily a normal subgroup of G.

2. Let K be the splitting field of $f(X) = X^3 - 2$ over $\mathbb{Q}$.
(a) Determine an explicit set of generators for K over $\mathbb{Q}$.
(b) Show that the Galois group $G(K/\mathbb{Q})$ of K over $\mathbb{Q}$ is isomorphic to the symmetric group S_3.
(c) Provide the complete list of intermediate fields k, $\mathbb{Q} \subseteq k \subseteq K$, satisfying $[k : \mathbb{Q}] = 3$.
(d) Which of the fields determined in (c) are normal extensions of $\mathbb{Q}$?

3. Calculate the complete character table for $\mathbb{Z}/3\mathbb{Z} \times S_3$, where S_3 is the symmetric group in 3 letters.

4. Let p be a prime number, $\mathbb{F}_p$ the prime field of p elements, X and Y algebraically independent variables over $\mathbb{F}_p$, $K = \mathbb{F}_p(X, Y)$, and $F = \mathbb{F}_p(X^p - X, Y^p - X)$.
(a) Show that $[K : F] = p^2$ and the separability and inseparability degrees of K/F are both equal to p.
(b) Show that there exists a field E, such that $F \subseteq E \subseteq K$, which is a purely inseparable extension of F of degree p.

5. (a) Prove that an $n \times n$ matrix A with entries in the field $\mathbb{C}$ of complex numbers, satisfying $A^3 = A$, can be diagonalized over $\mathbb{C}$.
(b) Does the statement in (a) remain true if one replaces $\mathbb{C}$ by an arbitrary algebraically closed field F ? Why or why not ?

6. Let R be the ring $\mathbb{Z}[X, Y] / \left(YX^2 - Y\right)$, where X and Y are two algebraically independent variables, and $\left(YX^2 - Y\right)$ is the ideal generated by $YX^2 - Y$

in $\mathbb{Z}[X,Y]$.
(a) Show that the ideal I generated by $Y-4$ in R is not prime.
(b) Provide the complete list of prime ideals in R containing the ideal I described in question (a).
(c) Which of the ideals found in (b) are maximal ?