PhD Qualifying Exam, Spring 2003, Algebra

1. Show that every group of order p^2, p a prime, is Abelian. Show that up to isomorphism there are only two such groups.

2. Let K be a field. A polynomial $f(x) \in K[x]$ is called *separable* if, in any field extension, it has distinct roots. Prove that:

(a) if K has characteristic 0, then each irreducible polynomial in $K[x]$ is separable; and

(b) if K has characteristic $p \neq 0$, then an irreducible polynomial $f(x) \in K[x]$ is separable if and only if has no form $g(x^p)$ where $g(x) \in K[x]$.

Give an example of an inseparable irreducible polynomial.

3. Prove that if a linear operator on a complex vector space is diagonal in some basis, then its restriction on any invariant subspace L is also diagonal in some basis of L.

4. A *differentiation* of a ring R is a mapping $D : R \to R$ such that, for all $x, y \in R$,

(1) $D(x + y) = D(x) + D(y)$; and

(2) $D(xy) = D(x)y + xD(y)$.

If K is a field and R is a K-algebra, then its differentiation are supposed to be over K, that is,

(3) $D(x) = 0$ for any $x \in K$.

Let D be a differentiation of the K-algebra $M_n(K)$ of $n \times n$-matrices. Find a matrix $A \in M_n(K)$ such that $D(X) = AX - XA$ for all $X \in M_n(K)$.

5. Prove the existence of a 1-dimensional invariant subspace for any 5-dimensional representation of the group A_4 (the *alternating* group of degree 4).