Algebra Qualifying Exam Spring 2004

May 17, 2004 (150 minutes)

Do all 6 problems. All problems are equally weighted. Show all details in your solutions.

Notation: $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are fields of rational, real and complex numbers.

1. Let $\mathbb{F}_2$ be the finite field with 2 elements.
(a) What is the order of $GL_3(\mathbb{F}_2)$, the group of 3×3 invertible matrices over $\mathbb{F}_2$?
(b) Assuming the fact that $GL_3(\mathbb{F}_2)$ is a simple group, find the number of elements of order 7 in $GL_3(\mathbb{F}_2)$.

2. Let $K \subset \mathbb{C}$ be the splitting field of $f(x) = x^6 + 3$ over $\mathbb{Q}$. Let α be a root of $f(x)$ in K.
(a) Show that $K = \mathbb{Q}(\alpha)$.
(b) Determine the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

3. Let k be a field with characteristic 0. Let $m \geq 2$ be an integer. Show that $f(x, y) = x^m + y^m + 1$ is irreducible in $k[x, y]$.

4. Let k be a field. Consider the integral domain $R = k[x, y]/(x^2 - y^2 + y^3)$.
(a) Show that R is not a unique factorization domain.
(b) Let F be the field of fractions of R. Find $t \in F$ such that $F = k(t)$.
(c) Determine the integral closure of R in F.

5. Show that there is a $\mathbb{C}$-algebra isomorphism between $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \times \mathbb{C}$.

6. Consider complex representations of a finite group G. Let $\sigma_1 \ldots \sigma_s$ be representatives from the conjugacy classes of G, and let $\chi_1 \ldots \chi_s$ be all the different simple characters of G.
(a). Define an inner product on the $\mathbb{C}$-space of class functions on G, so that $\{\chi_1 \ldots \chi_s\}$ forms an orthogonal basis for this space.
(b) Let $A = (a_{ij})$ be the matrix of the character table of G, i.e., $a_{ij} = \chi_i(\sigma_j)$ $(1 \leq i, j \leq s)$. Show that A is invertible.