Algebra Qualifying Exam   Spring 2005

May 23, 2005   (150 minutes)

Do all 6 problems. All problems are equally weighted. Show all details in your proofs.

1. Let k be a field. Let $G = GL_n(k)$ be the general linear group. Here $n > 0$. Let D be the subgroup of diagonal matrices. Let $N = N_G(D)$ be the normalizer of D. Determine the quotient group N/D.

2. Let $\mathbb{F}_p$ be the field with p elements, where p is a prime number. Let $f_{n,p}(x) = x^{p^n} - x + 1$, and suppose that $f_{n,p}(x)$ is irreducible in $\mathbb{F}_p[x]$. Let α be a root of $f_{n,p}(x)$.

(a) Show that $\mathbb{F}_{p^n} \subset \mathbb{F}_p(\alpha)$ and $[\mathbb{F}_p(\alpha) : \mathbb{F}_{p^n}] = p$.

(b) Determine all pairs (n, p) for which $f_{n,p}(x)$ is irreducible.

3. Let ξ be a primitive p^n-th root of unity. Here p is prime and $n > 0$. Let $f(x)$ be the minimal polynomial of ξ over $\mathbb{Q}$, and let m be its degree.

(a) Determine $f(x)$.

(b) Let $\alpha_1, \ldots, \alpha_m$ be all the roots of $f(x)$. Define the *discriminant* of ξ as:

$$D(\xi) = [\det(\alpha_i^{j-1})_{ij}]^2, \quad i, j = 1, \ldots m.$$

Show that $D(\xi) = (-1)^{\frac{m(m-1)}{2}} N_{\mathbb{Q}}^{\mathbb{Q}(\xi)}(f'(\xi))$.

(c) Take $n = 2$. Compute $D(\xi)$ in this case.

4. Let R be a ring. Let L be a minimal left ideal of R (i.e., L contains no nonzero proper left ideal of R). Assume $L^2 \neq 0$. Show that $L = Re$ for some non-zero idempotent $e \in R$.

5. Let A be an integral domain and let K be its field of fractions. Let A' be the integral closure of A in K. Let $P \subset A$ be a prime ideal and let $S = A - P$. (Note that $A_P = S^{-1}A$ is contained in K.) Show that A_P is integrally closed in K if and only if $(A'/A) \otimes_A A_P = 0$.

6. Let V be a finite dimensional vector space over a field k. Let G be a finite group. Let $\varphi : G \to GL(V)$ be an irreducible representation of G. Suppose that H is a finite abelian subgroup of $GL(V)$ such that H is contained in the centralizer of $\varphi(G)$. Show that H is cyclic.