Algebra Qualifying Exam    Spring 2006
May 24, 2006    (150 minutes)

Do all problems. All problems are equally weighted. Show all work.

1. Let $\mathbb{F}_p$ be the field with $p$ elements, here $p$ is prime. Let $SL_2(\mathbb{F}_p)$ be the group of 2x2 matrices over $\mathbb{F}_p$ with determinant 1.
   (1). Find the order of $SL_2(\mathbb{F}_p)$. Deduce that
   $H = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_p \right\}$
   is a Sylow-subgroup of $SL_2(\mathbb{F}_p)$.
   (2). Determine the normalizer of $H$ in $SL_2(\mathbb{F}_p)$ and find its order.

2. Let $R$ be a ring with identity 1. Let $x, y \in R$ such that $xy = 1$.
   (1). Assume $R$ has no zero-divisor. Show that $yx = 1$.
   (2). Assume $R$ is finite. Show that $yx = 1$.

3. Let $V$ be a n-dimensional vector space over a field $k$, with a basis $\{e_1, \ldots, e_n\}$. Let $A$ be the ring of all $n \times n$ diagonal matrices over $k$. $V$ is a $A$-module under the action:
   $\text{diag}(\lambda_1, \ldots, \lambda_n) \cdot (a_1 e_1 + \cdots + a_n e_n) = (\lambda_1 a_1 e_1 + \cdots + \lambda_n a_n e_n)$.
   Find all $A$-submodules of $V$.

4. Let $k$ be a field. Let $p$ be a prime number. Let $a \in k$. Show that the polynomial $x^p - a$ either has a root in $k$ or is irreducible in $k[x]$.

5. Let $V$ be a finite-dimensional vector space over a field $k$. Let $T \in End_k(V)$. Show that $tr(T \otimes T) = (tr(T))^2$. Here $tr(T)$ is the trace of $T$.

6. Let $S_4$ be the symmetric group of 4 elements.
   (1). Give an example of non-trivial 8-dimensional representation of the group $S_4$.
   (2). Show that for any 8-dimensional complex representation of $S_4$, there exists a 2-dimensional invariant subspace.