Algebra Qualifying Exam Spring 2006

May 24, 2006 (150 minutes)

Do all problems. All problems are equally weighted. Show all work.

1. Let $\mathbb{F}_p$ be the field with p elements, here p is prime. Let $SL_2(\mathbb{F}_p)$ be the group of 2x2 matrices over $\mathbb{F}_p$ with determinant 1.

(1). Find the order of $SL_2(\mathbb{F}_p)$. Deduce that

$$H = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_p\}$$

is a Sylow-subgroup of $SL_2(\mathbb{F}_p)$.

(2). Determine the normalizer of H in $SL_2(\mathbb{F}_p)$ and find its order.

2. Let R be a ring with identity 1. Let x, $y \in R$ such that $xy = 1$.

(1). Assume R has no zero-divisor. Show that $yx = 1$.

(2). Assume R is finite. Show that $yx = 1$.

3. Let V be a n-dimensional vector space over a field k, with a basis $\{e_1, \ldots, e_n\}$. Let A be the ring of all $n \times n$ diagonal matrices over k. V is a A-module under the action:

$$\text{diag}(\lambda_1, \ldots, \lambda_n) \cdot (a_1 e_1 + \cdots + a_n e_n) = (\lambda_1 a_1 e_1 + \cdots + \lambda_n a_n e_n).$$

Find all A-submodules of V.

4. Let k be a field. Let p be a prime number. Let $a \in k$. Show that the polynomial $x^p - a$ either has a root in k or is irreducible in $k[x]$.

5. Let V be a finite-dimensional vector space over a field k. Let $T \in End_k(V)$. Show that $tr(T \otimes T) = (tr(T))^2$. Here $tr(T)$ is the trace of T.

6. Let S_4 be the symmetric group of 4 elements.

(1). Give an example of non-trivial 8-dimensional representation of the group S_4.

(2). Show that for any 8-dimensional complex representation of S_4, there exists a 2-dimensional invariant subspace.