Algebra Qualifying Exam, Spring, 2007

May 14, 2007

1. Prove that the integer orthogonal group $O_n(\mathbf{Z})$ is a finite group. (By definition, an $n \times n$ square matrix X over $\mathbf{Z}$ is orthogonal if $XX^t = I_n$.)

2. Prove that no group of order 224 is simple.

3. Write down the irreducible polynomial for $\sqrt{2} + \sqrt{3}$ over $\mathbf{Q}$ and prove that it is reducible modulo p for every prime p.

4. Find the invertible elements, the zero divisors and the nilpotent elements in the following rings:

   (a) $\mathbf{Z}/p^n\mathbf{Z}$, where n is a natural number, p is a prime one.

   (b) the upper triangular matrices over a field.

5. Prove that the group $GL(2, \mathbf{C})$ does not contain a subgroup isomorphic to S_4.