Algebra Qualifying Exam   Spring 2008

May 19, 2008   (150 minutes)

Do all problems. All problems are equally weighted. Show all work.

1. Let k be a field. Consider the subgroup $B \subset GL_2(k)$ where

$$B = \{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in k,\ ad \neq 0 \}.$$

(a). Let Z be the center of $GL_2(k)$. Show that

$$\bigcap_{x \in GL_2(k)} x^{-1} B x = Z.$$

(b). Assume k is algebraically closed. Show that

$$\bigcup_{x \in GL_2(k)} x^{-1} B x = GL_2(k).$$

(c). Assume k is a finite field. Can the statement in (b) still be true?

2. Let ξ be a primitive 9-th root of unity. Find the minimal polynomial of $\xi + \xi^{-1}$ over $\mathbb{Q}$.

3. Let K be the splitting field of the polynomial $X^4 - 6X^2 - 1$ over $\mathbb{Q}$.
(a). Compute $\mathrm{Gal}(K/\mathbb{Q})$.
(b). Determine all intermediate fields that are Galois over $\mathbb{Q}$.

4. Let $V \cong \mathbb{C}^n$ be an n-dimensional complex vector space with the standard basis $e_1, \ldots, e_n$. Consider the permutation group action $S_n \times V \to V$ where $\sigma(e_i) = e_{\sigma(i)}$. Decompose V into simple $\mathbb{C}[S_n]$-modules.

5. Let k be a field of characteristic zero. Assume that E and F are algebraic extensions of k and both contained in a larger field L. Show that the k-algebra $E \otimes_k F$ has no nonzero nilpotent elements.

6. Give an example of non-isomorphic finite groups with same character table. Construct the character table in detail.