Do all problems. All problems are equally weighted. Show all work.

1. Let \( H \) and \( K \) be two solvable subgroups of a group \( G \) such that \( G = HK \).
   
   (a). Show that if either \( H \) or \( K \) is normal in \( G \), then \( G \) is solvable.
   
   (b). Give an example that \( G \) may not be solvable without the assumption in (a).

2. Consider \( \mathbb{Z}[^\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\} \) where \( \omega \) is a non-trivial cube root of 1. Show that \( \mathbb{Z}[^\omega] \) is an Euclidean domain.

3. Consider the field \( K = \mathbb{Q}(\sqrt[3]{a}) \) where \( a \in \mathbb{Z}, a < 0 \). Show that \( K \) cannot be embedded in a cyclic extension whose degree over \( \mathbb{Q} \) is divisible by 4.

4. Let \( E \) be a finite-dimensional vector space over an algebraically closed field \( k \). Let \( A, B \) be \( k \)-endomorphisms of \( E \). Assume \( AB = BA \). Show that \( A \) and \( B \) have a common eigenvector.

5. Consider the \( \mathbb{Z} \)-modules \( M_i = \mathbb{Z}/2^i\mathbb{Z} \) for all positive integers \( i \). Let \( M = \prod_{i=1}^{\infty} M_i \). Let \( S = \mathbb{Z} - \{0\} \).
   
   (a). Show that \( \mathbb{Q} \otimes_{\mathbb{Z}} M \cong S^{-1}M \).

Here \( S^{-1}M \) is the localization of \( M \).

(b). Show that \( \mathbb{Q} \otimes_{\mathbb{Z}} \prod_{i=1}^{\infty} M_i \neq \prod_{i=1}^{\infty} (\mathbb{Q} \otimes_{\mathbb{Z}} M_i) \).

6. Let \( G = S_4 \). Consider the subgroup \( H = \langle (12), (34) \rangle \).
   
   (a). How many simple characters over \( \mathbb{C} \) does \( H \) have?

(b). Choose a non-trivial simple character \( \psi \) of \( H \) over \( \mathbb{C} \) such that \( \psi((12)(34)) = -1 \). Computer the values of the induced character \( \text{ind}^G_H(\psi) \) on conjugacy classes of \( G \), then write the induced character as sum of simple characters.