Algebra Qualifying Exam   Spring 2009

May 14, 2009   (150 minutes)

Do all problems. All problems are equally weighted. Show all work.

1. Let H and K be two solvable subgroups of a group G such that $G = HK$.

(a). Show that if either H or K is normal in G, then G is solvable.

(b). Give an example that G may not be solvable without the assumption in (a).

2. Consider $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$ where ω is a non-trivial cube root of 1. Show that $\mathbb{Z}[\omega]$ is an Euclidean domain.

3. Consider the field $K = \mathbb{Q}(\sqrt{a}\,)$ where $a \in \mathbb{Z}$, $a < 0$. Show that K cannot be embedded in a cyclic extension whose degree over $\mathbb{Q}$ is divisible by 4.

4. Let E be a finite-dimensional vector space over an algebraically closed field k. Let A, B be k-endomorphisms of E. Assume $AB = BA$. Show that A and B have a common eigenvector.

5. Consider the $\mathbb{Z}$-modules $M_i = \mathbb{Z}/2^i\mathbb{Z}$ for all positive integers i. Let $M = \prod_{i=1}^{\infty} M_i$. Let $S = \mathbb{Z} - \{0\}$.

(a). Show that

$$\mathbb{Q} \otimes_{\mathbb{Z}} M \cong S^{-1}M.$$

Here $S^{-1}M$ is the localization of M.

(b). Show that

$$\mathbb{Q} \otimes_{\mathbb{Z}} \prod_{i=1}^{\infty} M_i \neq \prod_{i=1}^{\infty} (\mathbb{Q} \otimes_{\mathbb{Z}} M_i).$$

6. Let $G = S_4$. Consider the subgroup $H = <(12), (34)>$.

(a). How many simple characters over $\mathbb{C}$ does H have?

(b). Choose a non-trivial simple character ψ of H over $\mathbb{C}$ such that $\psi((12)(34)) = -1$. Computer the values of the induced character $\mathrm{ind}_H^G(\psi)$ on conjugacy classes of G, then write the induced character as sum of simple characters.