Algebra Qualifying Exam  Spring 2010

May 13, 2010  (150 minutes)

Do all problems. All problems are equally weighted. Show all work.

1. Let $G$ be a non-abelian group of order $p^3$, where $p$ is prime. Determine the number of distinct conjugacy classes in $G$.

2. Let $R$ be a ring such that $r^3 = r$ for all $r \in R$. Show that $R$ is commutative. (Hint: First show that $r^2$ is central for all $r \in R$.)

3. Compute Galois groups of the following polynomials.
   (a). $x^3 + t^2 x - t^3$ over $k$, where $k = \mathbb{C}(t)$ is the field of rational functions in one variable over complex numbers $\mathbb{C}$.
   (b). $x^4 - 14x^2 + 9$ over $\mathbb{Q}$.

4. Let $V$ be a $n$-dimensional vector space over a field $k$. Let $T \in \text{End}_k(V)$.
   (a). Show that $\text{tr}(T \otimes T \otimes T) = (\text{tr}(T))^3$. Here $\text{tr}(T)$ is the trace of $T$.
   (b). Find a similar formula for the determinant $\text{det}(T \otimes T \otimes T)$.

5. Classify all non-commutative semi-simple rings with 512 elements. (You can use the fact that finite division rings are fields.)