Algebra Qualifying Exam, May 2011

All problems are equally weighted. Explain clearly how you arrive at your solutions. You will be given three hours in which to complete the exam.

1. (a) Let $H$ be a subgroup of a finite group $G$ with $H \neq \{1\}$ and $H \neq G$. Prove that $G$ is not the union of all the conjugates of $H$ in $G$.

(b) Give an example of an infinite group $G$ for which the assertion in part (a) is false.

2. Let $p$ be a prime, $F$ a finite field with $p$ elements and $K$ a finite extension of $F$. Denote by $F^\times$ and $K^\times$ the multiplicative groups of nonzero elements of fields $F$ and $K$, respectively. Prove that the norm homomorphism $N : K^\times \to F^\times$ is surjective.

3. Determine the Galois group [up to isomorphism] of the splitting field of each of the following polynomials over $\mathbb{Q}$:

(a) $f(x) = x^4 - 9x^3 + 9x + 4$,

(b) $g(x) = x^5 - 6x^2 + 2$.

4. Let $F$ be a field, and $V$ a finite-dimensional vector space over $F$, with $\dim_F V = n$.

(a) Prove that if $n > 2$, the spaces $\bigwedge^2(\bigwedge^2(V))$ and $\bigwedge^4(V)$ are not isomorphic.

(b) Let $k$ be a positive integer. Prove that when $v \in \bigwedge^k(V)$ and $0 \neq x \in V$, $v \wedge x = 0$ holds if and only if $v = x \wedge y$ for some $y \in \bigwedge^{k-1}(V)$.

5. Let $K$ be a field, and $\Phi : G \to GL_n(K)$ an $n$-dimensional matrix representation of the group $G$. Define an action of $G$ on the full matrix ring $M_n(K)$ by $(g,A) \mapsto \Phi(g) \cdot A$ when $g \in G$ and $A \in M_n(K)$ (that’s a matrix product on the right-hand side). This in turn induces a group homomorphism $\Psi : G \to GL(M_n(K))$. Express the character $\chi(\Psi)$ of $\Psi$ in terms of $\chi(\Phi)$. 

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