Algebra Qualifying Exam, May 2011

All problems are equally weighted. Explain clearly how you arrive at your solutions. You will be given three *hours in which to complete the exam.*

1. (a) Let H be a subgroup of a finite group G with $H \neq \{1\}$ and $H \neq G$. Prove that G is not the union of all the conjugates of H in G.

   (b) Give an example of an infinite group G for which the assertion in part (a) is false.

2. Let p be a prime, F a finite field with p elements and K a finite extension of F. Denote by $F^\times$ and $K^\times$ the multiplicative groups of nonzero elements of fields F and K, respectively. Prove that the norm homomorphism $N : K^\times \to F^\times$ is surjective.

3. Determine the Galois group [up to isomorphism] of the splitting field of each of the following polynomials over $\mathbb{Q}$:

   (a) $f(x) = x^4 - 9x^3 + 9x + 4$,

   (b) $g(x) = x^5 - 6x^2 + 2$.

4. Let F be a field, and V a finite-dimensional vector space over F, with $\dim_F V = n$.

   (a) Prove that if $n > 2$, the spaces $\bigwedge^2(\bigwedge^2(V))$ and $\bigwedge^4(V)$ are not isomorphic.

   (b) Let k be a positive integer. Prove that when $v \in \bigwedge^k(V)$ and $0 \neq x \in V$, $v \wedge x = 0$ holds if and only if $v = x \wedge y$ for some $y \in \bigwedge^{k-1}(V)$.

5. Let K be a field, and $\Phi : G \to GL_n(K)$ an n-dimensional matrix representation of the group G. Define an action of G on the full matrix ring $M_n(K)$ by $(g, A) \mapsto \Phi(g) \cdot A$ when $g \in G$ and $A \in M_n(K)$ (that's a matrix product on the right-hand side). This in turn induces a group homomorphism $\Psi : G \to GL(M_n(K))$. Express the character $\chi(\Psi)$ of Ψ in terms of $\chi(\Phi)$.