Algebra Qualifying Exam, Spring 2012

All problems are equally weighted. Explain clearly how you arrive at your solutions. You will be given three hours in which to complete the exam.

1. Let G be a group of order p^3q^2, where p and q are prime integers. Show that for p sufficiently large and q fixed, G contains a normal subgroup other than $\{1\}$ and G.

2. (a) Prove that if M is an abelian group and n is a positive integer, the tensor product $M \otimes_{\mathbb{Z}} Z/n\mathbb{Z}$ can be naturally identified with M/nM.

   (b) Compute the tensor product over $\mathbb{Z}$ of $\mathbb{Z}/n\mathbb{Z}$ with each of $\mathbb{Z}/m\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$. Also compute the tensor products $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})$, and $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})$.

   (c) Let $\mathbb{Z}^{\mathbb{N}}$ denote the (abelian) group of sequences $(a_i)_{\imath\in\mathbb{N}}$ in $\mathbb{Z}$ under termwise addition, and $\mathbb{Z}^{(\mathbb{N})}$ the subgroup of sequences for which $a_i = 0$ for all but finitely many i. Define $\mathbb{Q}^{\mathbb{N}}$ and $\mathbb{Q}^{(\mathbb{N})}$ analogously. Compare $\mathbb{Z}^{(\mathbb{N})} \otimes_{\mathbb{Z}} \mathbb{Q}$ to $\mathbb{Q}^{(\mathbb{N})}$, and $\mathbb{Z}^{\mathbb{N}} \otimes_{\mathbb{Z}} \mathbb{Q}$ to $\mathbb{Q}^{\mathbb{N}}$.

3. In this problem, G denotes the group $S_5 \times C_2$, where S_5 is the symmetric group on five letters and C_2 is the cyclic group of order 2.

   (a) Determine all normal subgroups of G.

   (b) Give an example of a polynomial with rational coefficients whose Galois group is G, deducing that from basic principles.

4. Let Q denote the finite group of quaternions, with presentation

   $$Q = \{t, s_i, s_j, s_k \,|\, t^2 = 1,\ s_i^2 = s_j^2 = s_k^2 = s_i s_j s_k = t\}.$$

   (a) Determine four non-isomorphic representations of Q of dimension 1 over $\mathbb{R}$.

   (b) Show that the natural embedding of Q into the algebra $\mathbb{H}$ of real quaternions ($t \mapsto -1$, $s_i \mapsto i$, $s_j \mapsto j$, $s_k \mapsto k$) defines an irreducible real representation of Q, of dimension 4 over $\mathbb{R}$.

   (c) Determine all irreducible representations of Q over $\mathbb{C}$ (up to isomorphism).

5. (a) Give the definition of a Dedekind domain.

   (b) Give an example of a Dedekind domain that is not a principal ideal domain. Verify from the definition that it *is* a Dedekind domain, and also that it isn't a principal ideal domain.