Algebra Qualifying Exam
May 8, 2013

Student's Name:

To receive full credit, show all of your work.

1. Prove that, as a $\mathbb{Z}$-module, $\mathbb{Q}$ is flat but not projective.
2. Let p and q be primes with $p < q$. Let G be a group of order pq. Prove the following statements:
   (a) If p does not divide $q-1$ then G is cyclic.
   (b) If p divides $q-1$ then G is either cyclic or isomorphic to a non-abelian group on two generators. Give the presentation of this non-abelian group.
3. Prove that every group of order p^2q where p and q are primes is solvable.
4. Prove that the group of automorphisms $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{R})$ of the field $\mathbb{R}$ that fix $\mathbb{Q}$ pointwise is trivial (Hint: Prove that every such automorphism is continuous).
5. Let A and B be $n \times n$ matrices with complex coefficients. Assume that $(A-I)^n = 0$ and $A^kB = BA^k$ for some natural number k. Prove that $AB = BA$ (Hint: Prove that A can be expressed as a function of A^k).
6. Let K be the splitting field of $x^6 - 5$ over $\mathbb{Q}$.
   (a) Prove that $x^6 - 5$ is irreducible over $\mathbb{Q}$.
   (b) Compute the Galois group of K over $\mathbb{Q}$.
   (c) Describe an intermediate field F such that F is not $\mathbb{Q}$ or K and $F/\mathbb{Q}$ is Galois.