Algebra Qualifying Exam, Spring, 2015

May 13th, 12:30-3:30

1. Let K be a field of characteristic $p > 0$. Prove that a polynomial $f(x) = x^p - x - a \in K[x]$ either irreducible, or is a product of linear factors. Find this factorization if f has a root $x_0 \in K$.

2. Let A, B be two commuting operators on a finite dimensional space V over $\mathbb{C}$ such that $A^n = B^m$ is the identity operator on V for some positive integers n, m. Prove that V is a direct sum of 1-dimensional invariant subspaces with respect to A and B simultaneously.

3. Let G be a finite p-group. Determine all irreducible representations of G over a field K of characteristic p.

4. Prove that the polynomial $x^4 + 1$ is not irreducible over any field of positive characteristic.

5. Prove that a tensor product of irreducible representations over an algebraically closed field is irreducible.

6. Let D be an algebra over a field K of characteristic $\neq 2$ generated by two elements i, j satisfying relations

$$i^2 = 1\ , j^2 = 1\ , ij = -ji.$$

Prove, that $D \simeq GL_2(K)$.