JOHNS HOPKINS UNIVERSITY
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAMINATION
MAY 15, 2017, 12:00–3:00 PM

Each problem is worth 10 points.

(1) Let A be a commutative ring, and define the *nilradical* $\sqrt{0}$ to be the set of nilpotent elements in A. Show that $\sqrt{0}$ is equal to the intersection of all prime ideals in A. Show that if A is reduced, then A can be embedded into a product of fields.

(2) Write down the minimal polynomial for $\sqrt{2}+\sqrt{3}$ over $\mathbb{Q}$ and prove that it is reducible over $\mathbb{F}_p$ for every prime number p.

(3) Let K/k be a finite separable field extension, and let L/k be any field extension. Show that $K \otimes_k L$ is a product of fields.

(4) Let M be an invertible $n \times n$ matrix with entries in an algebraically closed field k of characteristic not 2. Show that M has a square root, i.e. there exists $N \in \mathrm{Mat}_{n\times n}(k)$ such that $N^2 = M$.

(5) Prove directly from the definition of (left) semisimple ring that every such ring is (left) Noetherian and Artinian. (You may freely use facts about semisimple, Noetherian, and Artinian modules.)

(6) Let G be a finite group and H an abelian subgroup. Show that every irreducible representation of G over $\mathbb{C}$ has dimension $\leq [G : H]$.