

JOHNS HOPKINS UNIVERSITY  
DEPARTMENT OF MATHEMATICS  
ALGEBRA QUALIFYING EXAMINATION  
SEPTEMBER 6, 2017, 12:30–3:30 PM

Each problem is worth 10 points.

- (1) Show that there is no simple group of order 30.
- (2) Let  $\Lambda$  be a free abelian group of finite rank  $n$ , and let  $\Lambda' \subset \Lambda$  be a subgroup of the same rank. Let  $x_1, \dots, x_n$  be a  $\mathbb{Z}$ -basis for  $\Lambda$ , and let  $x'_1, \dots, x'_n$  be a  $\mathbb{Z}$ -basis for  $\Lambda'$ . For each  $i$ , write  $x'_i = \sum_{j=1}^n a_{ij}x_j$ , and let  $A := (a_{ij}) \in \text{Mat}_{n \times n}(\mathbb{Z})$ . Show that the index  $[\Lambda : \Lambda']$  equals  $|\det A|$ .
- (3) In this problem all rings are commutative.
  - (a) Let  $R$  be a local ring with maximal ideal  $\mathfrak{m}$ , let  $N$  and  $M$  be finitely generated  $R$ -modules, and let  $f: N \rightarrow M$  be an  $R$ -linear map. Show that  $f$  is surjective if and only if the induced map  $N/\mathfrak{m}N \rightarrow M/\mathfrak{m}M$  is.
  - (b) Recall that a module  $M$  over a ring  $R$  is *projective* if the functor  $\text{Hom}_R(M, -)$  is exact. Show that if  $R$  is local and  $M$  is finitely generated projective, then  $M$  is free.
- (4) Compute the Galois group of  $x^5 - 10x + 5$  over  $\mathbb{Q}$ .
- (5) Let  $K/k$  be an extension of finite fields with  $\#k = q$ , let  $\Phi: x \mapsto x^q$  denote the  $q$ th power Frobenius map on  $K$ , and let  $G := \text{Gal}(K/k)$ .
  - (a) Compute the minimal polynomial of  $\Phi$  as a  $k$ -linear endomorphism of  $K$ .
  - (b) Use (a) to prove the *normal basis theorem* in the case of the extension  $K/k$ : there exists  $x \in K$  such that the set  $\{\sigma x \mid \sigma \in G\}$  is a  $k$ -basis for  $K$ .<sup>1</sup> (According to taste, it may be helpful to note that this is equivalent to the statement that  $K \simeq k[G]$  as  $k[G]$ -modules.)
- (6) Let  $G$  be a finite group with center  $Z \subset G$ . Show that if  $G$  admits a faithful irreducible representation  $G \rightarrow \text{GL}_n(k)$  for some positive integer  $n$  and some field  $k$ , then  $Z$  is cyclic.

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<sup>1</sup>In fact, the normal basis theorem holds for an arbitrary finite Galois field extension  $K/k$ , but a different proof is required when  $k$  is infinite.