ANALYSIS QUALIFYING EXAM, FALL 2003

Directions: This is a closed book exam. You have two hours to do all six of the (equally weighted) problems.

Question 1. Suppose that $f \in L^1(\mathbf{R})$. Prove that given $\epsilon > 0$, there exists $\delta > 0$ so that

$$\int_A |f| < \epsilon \text{ for every measurable set } A \text{ with } |A| < \delta\,,$$

where $|A|$ denotes the measure of A.

Question 2. Suppose that $f \in C^1([0, \pi]))$ and $f(0) = f(\pi) = 0$. Prove that

$$\int_0^\pi f^2 \leq \int_0^\pi (f')^2\,.$$

Question 3. Suppose that $1 < p < \infty$ and the linear mapping T is defined by

$$Tf(x) = x^{-1/p} \int_0^x f(t)\,dt\,.$$

Show that T is a bounded map from $L^q((0, \infty))$ to $C^0((0, \infty))$, where q satisfies $1/p + 1/q = 1$.

Question 4. Determine the number of zeros the function $f(z) = 2z^5 + 8z - 1$ has in the annulus $1 < |z| < 2$.

Question 5. Suppose that f is holomorphic on the punctured disk $0 < |z| < 2$.

(A) Prove that if there is a real constant C such that $|f(z)| \leq C$, then

$$\int_{|z|<1} |f'(z)|^2 dz < \infty\,.$$

(B) What happens when $|f|$ is unbounded?

Question 6. Suppose that $u > 0$ is a positive harmonic function on the punctured plane $0 < |z|$. Prove that u is constant.