ANALYSIS QUALIFYING EXAM, FALL 2004

Directions: This is a closed book exam. You have two and a half hours to do all seven problems. #7 is worth 10 points; the others are worth 20 points each.

1. a) Let $C([0,1])$ denote the space of continuous functions on [0,1], endowed with the "sup" norm. Show that $C([0,1])$ is a Banach space.

b) Let $B_p = L^p([0,1])$, with $1 < p < \infty$. Define weak and strong convergence in B_p. Then, show that the sequence $f_n(x) = \sin n\pi x$ converges weakly to 0, but *not* strongly to 0, in B_2.

2. a) Let f be integrable over a set A and suppose $A = \cup_{n=1}^{\infty} A_n$, where the A_n are pairwise disjoint. Show that

$$\int_A f = \sum_{n=1}^{\infty} \int_{A_n} f$$

and that the sum on the right-hand side is absolutely convergent.

b) Let μ be Lebesgue measure on $\mathbb{R}^2$ and let $f \in L^1(\mathbb{R}^2)$. Show there is a Borel measure λ for which $d\lambda = f d\mu$ (verify that it is a measure).

c) For $f = x^2 + y^2$ and D the unit disc, compute $\lambda(D)$.

3. Let $f \in L^1(\mathbb{R})$. Show directly (i.e., do not cite properties of the Fourier transform) that the function

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$$

is uniformly continuous and $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

4. Show that $f(x) = \dfrac{\cos x}{1+x^2}$ is an L^1 function on the real line (with respect to Lebesgue measure). Then evaluate

$$\int_0^{\infty} \frac{\cos x \; dx}{1+x^2}.$$

5. Determine whether the equation $z^3 + z^4 = 2$ in the complex variable z has any *non-real* solutions with $|z| < 2$.

6. Let f be an entire function with $|f(z)| \leq 3 \log |z|$ when $|z| > 2$. Either verify that f must be constant, or give a counterexample.

7. Let γ denote the curve $|z-1| = 2$, oriented counterclockwise. Evaluate

$$\int_{\gamma} \frac{e^z dz}{z^3}.$$