ANALYSIS QUALIFYING EXAM
SEPTEMBER 2005

Do all 8 problems. All problems are equally weighted. Time: 3 hours.

Show all work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

1. Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on $[0,1]$, and assume that
$$\int_0^1 |f_n(x)|^2 dx \le \frac{1}{n^2}.$$
Show that:
$$\lim_{n\to\infty} f_n(x) = 0 \quad \text{a.e. on} \quad [0,1]\ .$$

2. Let $f \in L^1(\mathbb{R}, dx)$. Prove that
$$\lim_{h\to 0} \int_{\mathbb{R}} |f(x+h) - f(x)| dx = 0.$$

3. Let g_n be a sequence of functions in $L^1(S^1, d\theta)$ where S^1 is the unit circle $\{e^{i\theta} : 0 \le \theta \le 2\pi\}$. We say that $g_n \to 0$ weakly if $\int_{S^1} g_n\left(e^{i\theta}\right) f\left(e^{i\theta}\right) d\theta \to 0$ as $n \to \infty$ for all $f \in C(S^1)$.

   Question: Suppose that $\{g_n\}$ is a sequence in $L^1(S^1, d\theta)$ and $\int_{S^1} e^{ik\theta} g_n\left(e^{i\theta}\right) d\theta \to 0$ as $n \to \infty$ for all $k \in \mathbb{Z}$. Need $g_n \to 0$ weakly? Give either a proof or a counterexample.

4. Suppose that $\{f_n\}$ is a sequence of elements of a Hilbert space X and that $f_n \to f$ weakly (i.e., $(f_n, g) \to (f, g)$ for all $g \in X$).

   (a) Show that
   $$\|f\| \le \liminf_{n\to\infty} \|f_n\|.$$
   Give an example showing that strict inequality can occur.

   (b) Suppose in addition that $\|f\| = \lim_{n\to\infty} \|f_n\|$. Show that $f_n \to f$ in norm.

5. Use contour integration to evaluate
$$\int_0^{+\infty} \frac{dx}{x^{1/3}(1+x)}.$$
Hint: Consider the contour beginning with the segment from ε to R, then traversing a circle of large radius R, then going back to ε, and finally traversing a circle of small radius ε.

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6. (a) Describe all the automorphisms of the upper half plane $H = \{z \in \mathbb{C} : \text{Re } z > 0\}$ (holomorphic bijective maps from H onto H).

   (b) Describe all the automorphisms of $\mathbb{C}$ (holomorphic bijective maps from $\mathbb{C}$ onto $\mathbb{C}$).

7. How many zeros does the polynomial

$$z^9 + z^5 - 8z^3 - z + 2$$

   have between the circles $\{|z| = 1\}$ and $\{|z| = 2\}$. Justify your answer.

8. Let $H = \{z \in \mathbb{C} : \text{Re } z > 0\}$ denote the upper half plane.

   (a) Does there exist a surjective holomorphic map $f : H \to \mathbb{C}$? Either give an example or prove that one does not exist.

   (b) Does there exist a surjective holomorphic map $f : \mathbb{C} \to H$? Either give an example or prove that one does not exist.