ANALYSIS QUALIFYING EXAM
SEPTEMBER 2006

Do all 8 problems. All problems are equally weighted. Time: 3 hours.

Show all work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Notation: $D = \{z \in \mathbb{C} : |z| < 1\}$

1. Use residues to calculate the integral

$$\int_0^\infty \frac{dx}{x^4 + 4} .$$

2. Let $f_n : D \to \mathbb{C}$, $n = 1, 2, 3, \ldots$, be a sequence of holomorphic functions on the unit disk D such that $f_n^{-1}(0) = \{c_n\}$, where $c_n \in D$. Suppose that $f_n \to f_0$ uniformly, where f_0 is not constant.

   a) Prove that f_0 has at most one zero in D.

   b) Can f_0 have no zeros? If so, give a necessary and sufficient condition on the c_n for this to happen.

3. State whether each of the following two statements is true or false, and give either a proof or counterexample for each.

   a) All holomorphic functions $f : \mathbb{C} \setminus \{0\} \to H$ are constant, where $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ denotes the upper half plane.

   b) All harmonic functions $h : \mathbb{C} \setminus [0, +\infty) \to [0, 1]$ are constant.

4. Let $f : D \to H$ be a holomorphic map from the unit disk D to the upper half plane $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

   Suppose that $f(0) = 3i$. Find the maximal possible value of $|f'(0)|$.

5. Let X be the Banach space of continuous real-valued functions on $[0, \pi]$ that vanish at 0 and π, equipped with the sup norm. Suppose that Y is a closed subspace of X where every element of Y can be written as a trigonometric polynomial, i.e., as a finite linear combination of the functions $\sin(kx)$ and $\cos(kx)$, for $k = 0, 1, 2, 3, \ldots$. Prove that Y is finite dimensional.

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6. Suppose that f is a C^1 function on $[0,2]$ and $f(0) = f'(0) = f(2) = f'(2) = 0$. Prove that for any $\varepsilon > 0$ there exists T_ε so that for all $t > T_\varepsilon$

$$\left| \int_0^2 f(x)\, e^{itx}\, dx \right| \leq \frac{\varepsilon}{t}\,.$$

7. Suppose that f_j is a sequence of L^2 functions on $[0,1]$ with

$$\int_0^1 |f_j| \leq 1/j \quad \text{and} \quad \int_0^1 f_j^2 \leq 1\,.$$

Prove that f_j goes to zero weakly in $L^2([0,1])$.

8. Suppose that X is a real Banach space and, for all $x, y \in X$, the norm $\|\cdot\|$ satisfies

$$\|x+y\|^2 + \|x-y\|^2 \leq 2\,\|x\|^2 + 2\,\|y\|^2\,.$$

Suppose also that $f : X \to \mathbb{R}$ is a linear functional with norm 1; that is,

$$\sup_{\|x\|=1} |f(x)| = 1\,.$$

Prove that there exists a unique point $x \in X$ with $\|x\| = 1$ and $f(x) = 1$.