Do all 8 problems. All problems are equally weighted. Time: 3 hours.

Show all work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Notation: \( D = \{ z \in \mathbb{C} : |z| < 1 \} \)

1. Use residues to calculate the integral
\[
\int_{0}^{\infty} \frac{dx}{x^4 + 4}.
\]

2. Let \( f_n : D \to \mathbb{C}, n = 1, 2, 3, \ldots \), be a sequence of holomorphic functions on the unit disk \( D \) such that \( f_n^{-1}(0) = \{ c_n \} \), where \( c_n \in D \). Suppose that \( f_n \to f_0 \) uniformly, where \( f_0 \) is not constant.
   a) Prove that \( f_0 \) has at most one zero in \( D \).
   b) Can \( f_0 \) have no zeros? If so, give a necessary and sufficient condition on the \( c_n \) for this to happen.

3. State whether each of the following two statements is true or false, and give either a proof or counterexample for each.
   a) All holomorphic functions \( f : \mathbb{C} \setminus \{ 0 \} \to H \) are constant, where \( H = \{ z \in \mathbb{C} : \text{Im } z > 0 \} \) denotes the upper half plane.
   b) All harmonic functions \( h : \mathbb{C} \setminus [0, +\infty) \to [0, 1] \) are constant.

4. Let \( f : D \to H \) be a holomorphic map from the unit disk \( D \) to the upper half plane \( H = \{ z \in \mathbb{C} : \text{Im } z > 0 \} \).
   Suppose that \( f(0) = 3i \). Find the maximal possible value of \( |f'(0)| \).

5. Let \( X \) be the Banach space of continuous real-valued functions on \([0, \pi]\) that vanish at 0 and \( \pi \), equipped with the sup norm. Suppose that \( Y \) is a closed subspace of \( X \) where every element of \( Y \) can be written as a trigonometric polynomial, i.e., as a finite linear combination of the functions \( \sin(kx) \) and \( \cos(kx) \), for \( k = 0, 1, 2, 3, \ldots \). Prove that \( Y \) is finite dimensional.

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6. Suppose that $f$ is a $C^1$ function on $[0, 2]$ and $f(0) = f'(0) = f(2) = f'(2) = 0$. Prove that for any $\varepsilon > 0$ there exists $T_\varepsilon$ so that for all $t > T_\varepsilon$

$$\left| \int_0^2 f(x) e^{itx} \, dx \right| \leq \frac{\varepsilon}{t}.$$ 

7. Suppose that $f_j$ is a sequence of $L^2$ functions on $[0, 1]$ with

$$\int_0^1 |f_j| \leq 1/j \quad \text{and} \quad \int_0^1 f_j^2 \leq 1.$$ 

Prove that $f_j$ goes to zero weakly in $L^2([0, 1])$.

8. Suppose that $X$ is a real Banach space and, for all $x, y \in X$, the norm $\| \cdot \|$ satisfies

$$\|x + y\|^2 + \|x - y\|^2 \leq 2\|x\|^2 + 2\|y\|^2.$$ 

Suppose also that $f : X \to \mathbb{R}$ is a linear functional with norm 1; that is,

$$\sup_{\|x\|=1} |f(x)| = 1.$$ 

Prove that there exists a unique point $x \in X$ with $\|x\| = 1$ and $f(x) = 1$. 

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