ANALYSIS QUALIFYING EXAM
FALL 2007

(1) Is the function
$$f(x,y) = x^3 + 3xy^2 - 3x^2y - 10 + i(y^3 + 3x^2y - 3y^2x + 5)$$
complex analytic? Prove that your answer is correct.

(2) Find all entire analytic functions satisfying $|f(z)| \leq |e^z|$ for all $z \in \mathbb{C}$.

(3) Let A be the annulus $A = \{z \in \mathbb{C} : \ 1 < |z| < 2\}$. Let f be a non-constant holomorphic function in a neighborhood of A, and suppose that $|f(z)| = 1$ on ∂A (the boundary of A). Prove that f has at least 2 zeros in A.

(4) Use the residue calculus to compute $\int_0^\infty \frac{dx}{1+x^n}$.

(5) Give examples of functions f and g on $\mathbb{R}$ so that $f \in L^1 \setminus L^2$ and $g \in L^2 \setminus L^1$.

(6) Does there exist an open dense subset of $\mathbb{R}$ with Lebesgue measure equal to one? Either construct an example or prove that one does not exist.

(7) Let f_n be a sequence of measurable real-valued functions on $[0,1]$ with
$$\sum_{n=1}^{\infty} \left(\int_0^1 |f_n| \right) \leq 1 \, .$$
Prove that f_n converges to zero almost everywhere.

(8) Suppose that f and g are $L^1(\mathbb{R})$ functions with compact support and let h be the convolution $f \star g$ (i.e., $h(x) = \int f(x-y)g(y)\,dy$). Prove that h is uniformly continuous.