PROBLEMS FOR ANALYSIS QUALIFYING EXAM
Fall 2008

Do all seven problems. Show all work and state any theorems you are using. Time: 3 hours.

1) (15 points) Consider the mapping $F : [0,1] \to [0,1]$ given by $F(s) = s^2$.

Let $F^{-j}(A)$ be the inverse image of j iterates of F applied to a measurable subset $A \subset [0,1]$. That is, if $F = F^1$ and F^j, $j = 2, 3, \ldots$ is defined inductively as $F^j = F^{j-1} \circ F$, then $F^{-j}(A) = \{x : F^j x = y, \text{ some } y \in A\}$.

a) Given $N = 1, 2, \ldots$ show that $\mu_N(A) = N^{-1} \sum_{j \le N} |F^{-j}(A)|$ is a measure which is absolutely continuous with respect to Lebesgue measure. Here $|B|$ denotes the Lebesgue measure of a measurable set.

b) Show that $\mu_N([a,b]) \to 0$ if $0 < a < b \le 1$.

c) If f is a continuous function on $[0,1]$ does $\lim \int_{[0,1]} f(s) d\mu_N(s)$ tend to a limit? If so, what is the limit?

2) (10 points) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and let $K(x,y)$ be a measurable function with respect to the product σ-algebra $\mathcal{M} \times \mathcal{N}$. Assume that there is a constant $0 < A < \infty$ so that for all $x \in X$

$$\int_Y |K(x,y)| \, d\nu(y) \le A,$$

and for all $y \in Y$,

$$\int_X |K(x,y)| \, d\mu(x) \le A.$$

Let $1 \le p \le \infty$ and for $f \in L^p(X, \mathcal{M}, \mu)$ define

$$Tf(y) = \int_X f(x) \, K(x,y) \, d\mu(x).$$

Prove that

$$\|TF\|_{L^p(\nu)} \le A \|f\|_{L^p(\mu)}.$$

3) (10 points) Is the Banach space ℓ^∞ of bounded complex sequences $a = \{a_n\}_{n=1}^\infty$ with the supremum norm $\|a\|_\infty = \sup_n |a_n|$ separable? Prove your assertion.

4) (10 points) Use residues to verify that

$$\int_0^\infty \frac{\ln x}{(x^2+4)^2} dx = \frac{\pi}{32}(\ln 2 - 1).$$

5) (10 points) How many solutions does the equation

$$e^z = 3z^7$$

have in the unit disk $D = \{x \in \mathbb{C} : |z| < 1\}$? Justify your answer.

6) (10 points) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Prove that if there exists some real number C and some positive integer k so that

$$|f(z)| \le C|z|^k$$

for all z with $|z| > 1$, then f is a polynomial in z of degree at most k.

7) (10 points) Let $D \subset \mathbb{C}$ be the unit disk and $\Omega \subset \mathbb{C}$ a bounded, simply connected domain. If $f_1 : D \to \Omega$ and $f_2 : D \to \Omega$ are holomorphic bijections so that $f_1(0) = f_2(0)$, then how are f_1 and f_2 related to each other?