ANALYSIS QUALIFYING EXAM
SEPTEMBER 2009

All problems are equally weighted. Show all your work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Time: 3 hours.

Part I. Real Analysis. Do 3 out of the following 4 problems.

1. Suppose f_n is a sequence of continuous functions on $[0,1]$ which converges to a continuous function f on $[0,1]$. Does it follow that f_n converge uniformly? Give a proof or provide a counterexample.

2. For which values of $\sigma \in \mathbb{R}$ does there exist a constant $C_\sigma < +\infty$ such that

$$\left| \sum_{j,k=1}^{\infty} (1+|j-k|)^{\sigma} a_j b_k \right| \leq C_\sigma \Big(\sum_{j=1}^{\infty} |a_j|^2 \Big)^{1/2} \Big(\sum_{k=1}^{\infty} |b_k|^2 \Big)^{1/2}$$

   Prove your assertion.

3. Let I be the unit interval $[0,1]$, and for $n = 1, 2, 3, \ldots$ and $0 \leq j \leq 2^n - 1$ let

$$I_{n,j} = [j2^{-n}, (j+1)2^{-n}].$$

   For $f \in L^1(I, dx)$ define $E_n f(x) = \sum_{j=0}^{2^n-1} \big(2^n \int_{I_{n,j}} f\, dt \big) \chi_{I_{n,j}}(x)$, where $\chi_{I_{n,j}}$ is the characteristic function of $I_{n,j}$. Prove that if $f \in L^1(I, dx)$ then $\lim_{n\to\infty} E_n f(x) = f(x)$ almost everywhere in I.

4. Let $f(x)$ be a non-decreasing function on $[0,1]$. You may assume that f is differentiable almost everywhere. Prove that

$$\int_0^1 f'(x)\, dx \leq f(1) - f(0).$$

Part II. Complex Analysis. Do 3 out of the following 4 problems.

5. Let

$$f(x+iy) = x^3 - 3xy^2 + iy^3 \,.$$

State whether each of the following is true or false and give proofs for your answers:

a) the complex derivative $f'(0)$ exists;

b) f is holomorphic in a neighborhood of 0.

6. Let

$$f(z) = \frac{z}{\tan z} \qquad \text{for} \quad z \neq 0.$$

a) Prove that f has a removable singularity at 0.

b) What is the radius of convergence of the power series for f centered at 0? Justify your answer.

7. Let $f : H \to D$ be a holomorphic map from the upper half plane $H = \{z \in \mathbb{C} : \text{Im}\, z > 0\}$ to the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$.

Suppose that $f(i) = 1/2$. Determine the maximal possible value of $|f'(i)|$.

8. Let h be a harmonic function on the punctured disk

$$U := \{z \in \mathbb{C} : 0 < |z| < 1\} \,.$$

Show that there exists a constant $c \in \mathbb{R}$ and a holomorphic function f on U such that $\text{Re} f(z) = h(z) + c \log |z|$ for all $z \in U$.