Analysis Qualifying Exam Fall 2010

Instructions: *Do all eight problems. Each problem will be scored out of 10 points.*

1. Suppose that $f_j \in L^2(\mathbb{R}^n)$, $j = 1, 2, 3, \ldots$ and that $f_j \to f$ in L^2. Suppose further that there is a constant $M < \infty$ so that

$$\int e^{100|x|^2} |f_j(x)|^2 \, dx \le M, \quad j = 1, 2, 3, \ldots.$$

Is it true that $\int e^{99|x|^2} |f(x)|^2 \, dx < \infty$? Give a proof or counterexample.

2. Let $E, F \subset \mathbb{R}$ be two Lebesgue-measurable subsets of $\mathbb{R}$, each of finite measure, and let χ_E and χ_F denote their respective characteristic functions.

(a) Prove that the convolution $\chi_E * \chi_F$ defined by

$$\chi_E * \chi_F(x) = \int_{\mathbb{R}} \chi_E(y) \chi_F(x - y) \, dy$$

is a continuous function of x.

(b) Show that as $n \to \infty$,

$$n\big(\chi_E * \chi_{[0,1/n]}\big) \to \chi_E$$

pointwise almost everywhere.

3. Let $Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \, dy$, where $K(x, y)$ is a nonnegative measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. Suppose that there are measurable functions $p(x) > 0$ and $q(x) > 0$ on $\mathbb{R}^n$ and real numbers $\alpha, \beta > 0$ for which

$$\int K(x, y) q(y) \, dy \le \alpha p(x),$$

for almost all x and

$$\int p(x) K(x, y) \, dx \le \beta q(y)$$

for almost all y. Show that for $f \in L^2(\mathbb{R}^n)$ we have

$$\|Tf\|_{L^2} \le \sqrt{\alpha\beta}\, \|f\|_{L^2}.$$

(This is called *Schur's test.*)

4. Define $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$Uf(x) = f(x - 1).$$

Show that if $f \in L^2$ satisfies $Uf = \lambda f$, for some $\lambda \in \mathbb{R}$ (i.e., f is an eigenvector of U) then f must be the zero element, i.e., $f = 0$ almost everywhere.

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5. Let γ be the closed curve in the complex plane that is given in polar coordinates by $r = 2 + 3\cos\theta,\ 0 \leq \theta \leq 4\pi$, oriented in the direction of increasing θ. Let

$$f(z) = \frac{e^z}{2z-1} + \frac{\sin(2z)}{(z-2)^2} + \frac{\cos(5z)}{(z+5i)^3}.$$

Calculate $\displaystyle\int_\gamma f(z)dz$.

[Recall that in polar coordinates, $(-r, \theta)$ and $(r, \theta + \pi)$ give the same point in the plane.]

6. Let D denote the open unit disc in $\mathbb{C}$. Let $f : D \to \mathbb{C}$ be a C^1 function, and consider the property: *f has a double zero at $z = \frac{1}{n}$ for all natural numbers n.*

(a) Determine all holomorphic functions f with this property. [The terms "holomorphic" and "complex analytic" have the same meaning.]

(b) Give an example of a *non*-holomorphic C^1 function with this property. *(You must explain why your example has this property.)*

7. Determine all entire functions f (i.e., $f(z)$ is holomorphic and is defined for all $z \in \mathbb{C}$) that satisfy the inequality:

$$|f(z)| \leq |\mathrm{Re}\, z|^2 + |z|^{\frac{3}{2}} \quad \text{whenever } |z| > 1.$$

8. Let D denote the open unit disc, as in #6. Let $g : D \to D$ be a surjective holomorphic mapping for which $g(0) = 0$. Suppose that $z = g(w)$ gives a two-sheeted branched covering of the image with exactly one branch point at $w = 0$. An example of such a function g is $g(w) = w^2$.

(a) Express the given conditions explicitly in terms of g and its derivatives.

(b) Show that $|g(w)| \leq |w|^2$ for all $|w| < 1$.

(c) Suppose that $g(1/2) = i/4$. What is the strongest statement about $g(w)$ that follows from the assertion in (b)? **Explain.**