ANALYSIS QUALIFYING EXAM
SEPTEMBER 2011

All problems are equally weighted. Show all your work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Time: 3 hours.

Part I. Complex Analysis.

Notation: $D = \{z \in \mathbb{C} : |z| < 1\}$

1. Determine the value of the integral

$$\int_\gamma \frac{dz}{z^3 \cos z},$$

where γ is the circle $\{|z - 1| < 2\}$ traversed counterclockwise.

2. Let $h : \mathbb{C} \to \mathbb{R}$ be a harmonic function such that h is bounded below. Prove that h is constant.

3. Let f be a holomorphic function on $D \setminus \{0\}$. Suppose that there exists a positive integer n such that $f^{-1}(w)$ contains at most n points for all $w \in \mathbb{C}$. Prove that 0 is a removable singularity or pole.

4. Suppose that U is a simply connected bounded domain in $\mathbb{C}$, and let $P \in U$. Prove that for all $t \in \mathbb{R}$, there exists a unique holomorphic function $f : U \to U$ such that $f(P) = P$ and $f'(P) = e^{it}$.

Part II. Real Analysis.

Notation: $|A|$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$.

5. Give an example of a sequence of functions $\{f_j\}$ satisfying $\|f_j\|_{L^2(\mathbb{R})} = 1$ for which $\{f_j\}$ has no convergent subsequence in $L^2(\mathbb{R})$.

6. a) Let $f_j \in L^2(\mathbb{R}^n)$ and suppose that

$$\int_{\mathbb{R}^n} |f_j(x) - f(x)|^2 \, dx \to 0.$$

If $\Omega \subset \mathbb{R}^n$ has finite Lebesgue measure, i.e., $|\Omega| < \infty$, show that the Fourier transforms satisfy

$$\int_{\Omega} \widehat{f_j}(\xi) \, d\xi \to \int_{\Omega} \widehat{f}(\xi) \, d\xi. \tag{1}$$

b) If $|\Omega| = \infty$, is (1) still always valid? Give a proof or counterexample.

7. Let $\omega(\alpha) = |\{x : |f(x)| > \alpha\}|$, $\alpha > 0$, be the distribution function of a given $f \in L^p(\mathbb{R}^n)$, where $p > 0$. Does $\alpha^p \omega(\alpha)$ tend to a limit as $\alpha \to 0+$? Give a proof or counterexample.

8. Show that there does not exist a function $I \in L^1(\mathbb{R}^n)$ such that

$$f * I = f \qquad \text{for all } f \in L^1(\mathbb{R}^n).$$

(Here $(f * I)(x) = \int f(y) \, I(x - y) \, dy$ is the convolution of f and I.)