All problems are equally weighted. Show all your work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Time: 3 hours.

Part I. Complex Analysis.

Notation: $D = \{ z \in \mathbb{C} : |z| < 1 \}$

1. Determine the value of the integral

$$\int_{\gamma} \frac{dz}{z^3 \cos z},$$

where $\gamma$ is the circle $\{|z - 1| < 2\}$ traversed counterclockwise.

2. Let $h : \mathbb{C} \to \mathbb{R}$ be a harmonic function such that $h$ is bounded below. Prove that $h$ is constant.

3. Let $f$ be a holomorphic function on $D \setminus \{0\}$. Suppose that there exists a positive integer $n$ such that $f^{-1}(w)$ contains at most $n$ points for all $w \in \mathbb{C}$. Prove that 0 is a removable singularity or pole.

4. Suppose that $U$ is a simply connected bounded domain in $\mathbb{C}$, and let $P \in U$. Prove that for all $t \in \mathbb{R}$, there exists a unique holomorphic function $f : U \to U$ such that $f(P) = P$ and $f'(P) = e^{it}$. 
Part II. Real Analysis.

Notation: \(|A|\) denotes the Lebesgue measure of a measurable set \(A \subset \mathbb{R}^n\).

5. Give an example of a sequence of functions \(\{f_j\}\) satisfying \(\|f_j\|_{L^2(\mathbb{R})} = 1\) for which \(\{f_j\}\) has no convergent subsequence in \(L^2(\mathbb{R})\).

6. a) Let \(f_j \in L^2(\mathbb{R}^n)\) and suppose that

\[
\int_{\mathbb{R}^n} |f_j(x) - f(x)|^2 \, dx \to 0.
\]

If \(\Omega \subset \mathbb{R}^n\) has finite Lebesgue measure, i.e., \(|\Omega| < \infty\), show that the Fourier transforms satisfy

\[
\int_{\Omega} \hat{f}_j(\xi) \, d\xi \to \int_{\Omega} \hat{f}(\xi) \, d\xi. \tag{1}
\]

b) If \(|\Omega| = \infty\), is (1) still always valid? Give a proof or counterexample.

7. Let \(\omega(\alpha) = |\{x : |f(x)| > \alpha\}|, \alpha > 0\), be the distribution function of a given \(f \in L^p(\mathbb{R}^n)\), where \(p > 0\). Does \(\alpha^p \omega(\alpha)\) tend to a limit as \(\alpha \to 0^+\)? Give a proof or counterexample.

8. Show that there does not exist a function \(I \in L^1(\mathbb{R}^n)\) such that

\[ f \ast I = f \quad \text{for all } f \in L^1(\mathbb{R}^n). \]

(Here \((f \ast I)(x) = \int f(y) I(x-y) \, dy\) is the convolution of \(f\) and \(I\).)