# ANALYSIS QUALIFYING EXAM, FALL 2012

## Part I. Complex Analysis.

1. How many zeros does the polynomial

$$z^9 + z^6 + 30\,z^5 - 3\,z + 2$$

   have in the annulus $\{1 \le |z| \le 3\}$. Justify your answer.

2. Let $f(x) = \frac{1}{x^2+1}$. Use residues to compute the Fourier transform

$$\widehat{f}(t) = \int_{-\infty}^{+\infty} f(x) e^{-itx}\,dx\ .$$

3. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk.

   What is the maximum possible value of $|f'(\frac{1}{2})|$ for a holomorphic function $f : D \to D$ with $f(\frac{1}{2}) = \frac{3}{4}$? Find all such functions $f$ that attain this maximum value.

4. Let $I = \{t \in \mathbb{R} : 0 \le t \le 1\} \subset \mathbb{C}$. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a continuous function such that $f$ is holomorphic on $\mathbb{C} \setminus I$. Prove that $f$ is an entire function (i.e., $f$ is holomorphic on all of $\mathbb{C}$).

**Part II. Real Analysis.**

5. For each natural number $n$, let $f_n : [0,1] \to \mathbb{R}$ be a sequence of absolutely integrable functions, and let $f : [0,1] \to \mathbb{R}$ be another absolutely integrable function such that

$$\int_0^1 \big|f_n(x) - f(x)\big| dx \to 0, \qquad \text{as} \quad n \to \infty.$$

(a) Show that there exists a subsequence $f_{n_j}$ of $f_n$ which converges to $f$ pointwise almost everywhere.

(b) Give a counterexample to show that the assertion fails if "pointwise almost everywhere" is replaced by "uniformly".

6. For this problem, consider just Lebesgue measurable functions $f : [0,1] \to \mathbb{R}$. together with the Lebesgue measure.

(a) State Fatou's lemma (no proof required).

(b) State and prove the Dominated Convergence Theorem.

(c) Give an example where $f_n(x) \to 0$ a.e., but $\int_{-\infty}^{+\infty} f_n(x)dx \to 1$.

7. Let

$$f * g(x) := \int_{-\infty}^{+\infty} f(y)g(x-y)dy$$

denote the convolution of $f$ and $g$.

(a) Let $f, g \in L^2(\mathbb{R})$ be two square-integrable functions on $\mathbb{R}$ (with the usual Lebesgue measure). Show that the convolution $f * g$ bounded continuous function on $\mathbb{R}$.

(b) Instead let $h \in L^1(\mathbb{R})$ be fixed. Show that $A(f) = f * h$ is a bounded operator $L^1(\mathbb{R}) \to L^1(\mathbb{R})$.

8. Let T be a linear transformation on $C_0(\mathbb{R})$, the space of continuous functions of compact support, that has the following two properties:

$$\|Tf\|_{L^\infty} \le \|f\|_{L^\infty}, \qquad \text{and} \qquad \big|\{x \in \mathbb{R} : |Tf(x)| > \lambda\}\big| \le \frac{\|f\|_{L^1}}{\lambda}.$$

(Here $|A|$ denotes the Lebesgue measure of the set $A$.) Prove that

$$\int_{-\infty}^{+\infty} |Tf(x)|^2 \, dx \le C \int_{-\infty}^{+\infty} |f(x)|^2 dx$$

for all $f \in C_0(\mathbb{R})$ and some fixed number $C$.