QUALIFYING EXAM - ANALYSIS - FALL 2013

Justify your answers to all problems.

1. Let I denote the interval $(0,1)$. Suppose that $f : I \to \mathbb{R}$ with $\int_0^1 |f(t)|\,dt < +\infty$. Define $g : I \to \mathbb{R}$ by

$$g(x) = \int_x^1 \frac{f(t)}{t}\,dt.$$

Show that $g \in L^1(I)$.

2. Does there exist a nonempty measurable set $E \subset \mathbb{R}$ satisfying the following two properties:

   (a) given $x, y \in E$, there exists $z \notin E$ that lies between x and y;

   (b) E has no isolated points?

3. Prove that smooth compactly supported functions are dense in $L^2(\mathbb{R}^n)$.

4. Determine whether there is a nonzero smooth compactly supported function on $\mathbb{R}$ whose Fourier transform is also compactly supported?

5. This problem is about the integral

$$I = \int_0^\infty \frac{\cos u\,du}{u^4+1}.$$

   (a) Show directly that I is a convergent improper Riemann integral.

   (b) Is

$$\int_{[0,\infty)} \frac{\cos u}{u^4+1}\,d\mu(u)$$

   a well-defined Lebesgue integral, where μ denotes the Lebesgue measure on $\mathbb{R}$?

   (c) (main part) Evaluate the integral in (a).

6. Determine the number of *distinct* solutions of the equation

$$e^{z^2} = 5z^5$$

in the unit disk $\{z \in \mathbb{C} : |z| \le 1\}$.

7. Determine all entire functions f (i.e., $f(z)$ is holomorphic on the whole z-plane) that satisfy the inequality

$$|f(z)| \le |z|^2 |\mathrm{Im}\, z|^2$$

for $|z|$ sufficiently large.