Qualifying Exam - Analysis - Fall 2014

Justify your answers to all problems.

1. Let Q be the unit square in $\mathbb{R}^2$. Consider functions $f_n \in L^1(Q)$ such that

$$f_n \to f \text{ almost everywhere in } Q \quad \text{and} \quad \int_Q |f_n| \to \int_Q |f| < \infty.$$

(a) Prove that $\int_A |f_n| \to \int_A |f|$ for every measurable subset A of Q.
(b) Prove that $f_n \to f$ in L^1.

2. Let $f \in L^1(\mathbb{R}^d)$ and M_f denote the Hardy-Littlewood maximal function of f; in other words,

$$M_f(x) = \sup_B \frac{1}{m(B)} \int_B |f(y)|dy, \quad x \in \mathbb{R}^d$$

where the supremum is taken over all balls containing the point x. Prove that

$$m\big(\{x : M_f(x) > \alpha\}\big) \leq \frac{A}{\alpha}||f||_{L^1(\mathbb{R}^d)}, \quad \forall \alpha > 0$$

where A is a constant depending only on d and $||f||_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f(x)|dx$.

3. Let X and Y be Hilbert spaces and $L : X \to Y$ be a bounded linear operator. Prove that the following two conditions are equivalent:

(a) The image $L(\mathbf{B})$ of the unit ball in X has compact closure in Y.
(b) There is a sequence of bounded linear operators $\{L_n : X \to Y\}$ such that the image of $L_n(X)$ is finite dimensional and such that $||L_n - L|| \to 0$. (Here, $||\cdot||$ is the operator norm.)

4. Let $\Omega \subset \mathbb{C}$ be a bounded region and $\{f_n\}$ be a sequence of continuous functions on $\overline{\Omega}$ which are holomorphic in Ω. If $\{f_n\}$ converges uniformly on the boundary of Ω, then prove that f_n converges uniformly on Ω.

5. Compute

$$\int_0^\infty \frac{\cos ax}{(1+x^2)^2} dx = \frac{\pi(a+1)e^{-a}}{4}, \quad a > 0.$$

6. Assume that f and g are entire functions and that g never vanishes. If $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$, then prove that there is a constant C such that $f(z) = Cg(z)$.

7. Let $D \subset \mathbb{C}$ be the unit disk. Prove that every one-to-one conformal mapping of D to D is given by a linear fractional transformation.