QUALIFYING EXAM - ANALYSIS - FALL 2015

Justify your answers to all problems.

Notation: $\mathbb{R}$ is the real line, $\mathbb{C}$ is the complex plane and $D(P,r) \subset \mathbb{C}$ is the disk of radius r centered at point P.

1. Suppose $\{f_n\}_{n=1}^{\infty} \subset L^2(\mathbb{R})$ is a sequence that converges to 0 in the L^2 norm; in other words,

$$||f_n||_{L^2(\mathbb{R})} = \left(\int_{-\infty}^{\infty} |f_n|^2 \, dx\right)^{\frac{1}{2}} \to 0.$$

Prove that there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to 0$ almost everywhere.

2. Determine whether the following statements are true and false. If true, provide a proof. If false, prove a counter example.

(a) If $f(x)$ is a increasing, continuous function on the interval $[0,1]$ such that $f(0) = 0$ and $f(1) = 1$, then there exists a set $E \subset [0,1]$ of positive measure such that $f'(x) > 0$.

(b) If $f(x)$ is a strictly increasing, absolutely continuous function on the interval $[0,1]$ with $f(0) = 0$ and $f(1) = 1$, then the set $f^{-1}(E) \cap \{x \in [0,1] : f'(x) > 0\}$ is measurable for any measurable set $E \subset [0,1]$.

3. Let $\{\varphi_k\}_{k=1}^{\infty}$ be an orthonormal basis for $L^2(\mathbb{R}^d)$ and define $\varphi_{k,j}(x,y) = \varphi_k(x)\varphi_j(y)$. Prove that $\{\varphi_{k,j}\}_{k,j=1}^{\infty}$ is an orthonormal basis of $L^2(\mathbb{R}^d \times \mathbb{R}^d)$.

4. Let $U \subset \mathbb{C}$ be an open set containing $\overline{D}(P,r)$. Prove that if $f : U \to \mathbb{C}$ is a holomorphic function such that f is nowhere zero on $\partial D(P,r)$ and $g : U \to \mathbb{C}$ is a holomorphic function sufficiently uniformly close to f on $\partial D(P,r)$, then the number of zeros of f in $D(P,r)$ equals the number of zeros of g in $D(P,r)$ (counting multiplicity).

5. If $f = u + iv$ is an entire function with the property that $u(z) \leq 0$ for all $z \in \mathbb{C}$, what can you say about f?

6. If $D(0,1) \to \mathbb{C}$ is a function such that f^2 and f^3 are both holomorphic, prove f is holomorphic.

7. Compute the integral

$$\int_0^{\infty} \frac{(\log x)^2}{1+x^2} \, dx.$$